NGR-05-007-116

Linear and Nonlinear Theory of Grid Excitation of Low Frequency Waves in a Plasma

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April, 1967 R-13

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This research was partially sponsored by the Office of Naval Research, NONR 4756(01).

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ACKNOWLEDGMENT

The patient encouragement and unsparing guidance of Professor Alfredo Baños, Jr., in this dissertation and in the education of its author are warmly and gratefully acknowledged.

The author also expresses his appreciation to Professor Burton D. Fried for his important contributions to this dissertation.

This work was supported by the U. S. Office of Naval Research, the National Science Foundation, the Ford Foundation, the University of California, Los Angeles, and the National Aeronautics and Space Administration. The facilitation of a difficult computational task by the TRW Systems On-Line Computer is acknowledged.

ABSTRACT OF THE DISSERTATION

Linear and Nonlinear Theory of Grid Excitation

of Low Frequency Waves in a Plasma

by

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Doctor of Philosophy in Physics

University of California, Los Angeles, 1967

Professor Alfredo Baños, Jr., Chairman

The steady-state response of an infinite, uniform expanse of hot, rarefied plasma to low-frequency excitation by a pair of idealized parallel plane grids is investigated. The Vlasov equation for grid excitation of longitudinal waves is considered both in the linearized (infinitesimal amplitude) limit and in the weakly nonlinear case of small but finite amplitude disturbances.

In the linearized theory, the Fourier inversion integral for the spatial behavior of the potential excited in the plasma by a pair of grids with finite spacing is evaluated by first transforming the variable of integration from the Fourier transform variable k to $\zeta = \omega_o/ka_i$, where ω_o is the driving frequency and a_i is the ion thermal speed, and then separating the integrand into two parts, one principally responsible for the ion acoustic wave near the grid and the other for the electron wave at large distances from the grid. For driving frequencies small in comparison with the ion plasma frequency,

the resulting integrals are evaluated numerically along appropriate deformed contours of integration. The path of steepest descents is chosen for the ion integral; it includes a residue contribution from the pole which gives the dominant behavior of the ion wave. In both cases, the integrands undergo small phase changes over the chosen contours. Numerical results are presented; in the dipole limit, the results of Gould[1] are recovered.

A perturbation series expansion of the potential and of the species distribution functions in the (nonlinear) Vlasov equation yields a hierarchy of equations associated with a smallness parameter proportional to the amplitude of excitation. In each order the equations are linear in the perturbation quantities of that order and have driving terms composed of quadratic combinations of lower order quantities. For sufficiently small amplitudes of excitation, the principal contributions to the response come from the first and second order equations. In the first order, the linearized Vlasov equation is obtained. In the second order, the steady-state response consists of zero frequency and double frequency components. In the manner of Landau, the second order equations are Laplace-Fourier transformed and resulting velocity integrals are analytically continued and expressed in terms of the plasma dispersion function. Confining our attention to driving frequencies small in comparison with the ion plasma frequency, and approximating the driving terms (linearized response) by their dominant pole component, we obtain the Fourier inversion integrals for the steady-state response. The integrals are evaluated numerically by methods developed in treating the linearized problem. The double frequency component of the response is strongly damped, like an "ion wave" but has a slow phase variation with distance, like an "electron wave". The zero frequency component is a polarization of the plasma unaccompanied by species current densities.

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1. R. W. Gould, Physical Review 136, A991 (1964).

I. INTRODUCTION

The existing theory of waves in plasmas deals extensively with the evolution in time of an initial perturbation in a plasma. Present in much of the existing theory is the assumption of small disturbances, which permits a linearized treatment. This dissertation considers the less familiar problem of the steady-state response of a plasma to a localized time-harmonic excitation. The dissertation treats not only the linearized case but also the case of the weakly nonlinear response to a level of excitation somewhat higher than that for which a linearized treatment is appropriate.

Section 1 of this chapter contains a specific statement of the problem. Section 2 contains a description of the Vlasov equation, which embodies the kinetic theory formulation used in this dissertation. Section 3 describes the Laplace-Fourier transform method of treating the linearized Vlasov equation and presents the Fourier inversion integral for the steady-state response of the plasma to excitation by a localized external charge density in the linearized theory. In Section 4 the analytic continuation of certain functions of a complex variable which appear in the transform of the potential is obtained as a necessary preliminary to the evaluation of the Fourier inversion integral for the steady-state response.

Chapter II describes a novel method of numerical evaluation of the Fourier inversion integral for the steady-state behavior of the potential in the linearized theory. The method is applied to the case of a distribution of external charge density different from that previously considered. Chapter III describes a perturbation series method of treating the weakly nonlinear response of the plasma to a level of excitation somewhat higher than that for which a linearized treatment is appropriate. The method is applied, by making a necessary approximation, to the present problem. One component of the lowest order nonlinear response, at zero frequency, is determined. Chapter IV contains the treatment of the other component of the lowest order nonlinear response, at twice the frequency of the external excitation.

1.1 Statement of the Problem

We consider the steady-state excitation of longitudinal waves in an infinite, uniform expanse of plasma composed of electrons and singly charged ions. The plasma considered is sufficiently hot and rarefied that the effects of close interactions between particles are negligible relative to the effects of collective interactions. Under these conditions the plasma may be described by the Vlasov or correlationless kinetic equation, which determines the evolution of the one-body distribution functions of ions and electrons.[1]

The source of excitation is a pair of idealized parallel plane grids which, in the steady state, establishes a one-dimensional, time-harmonic potential in the plasma; the grids are assumed not to intercept any particles of the plasma. The frequency of excitation is below the ion plasma frequency; the response accordingly involves both electron and ion dynamics. The response is damped with increasing distance from the grids. This behavior is the spatial analogue of the Landau damping in time of an initial perturbation in species distribution functions predicted from the Vlasov equation.[2]

The steady-state response to small amplitude grid excitation* described by the linearized Vlasov equation may be determined by numerical evaluation of a Fourier inversion integral. This evaluation has been performed by folding the primitive Fourier inversion contour to obtain a branch-cut integral along the positive real axis and by numerical integration along the deformed contour.[4] An alternative method of evaluating the integral, which possesses advantages over the method described, is presented in this dissertation, below. The new method is used to study the effect of variable distance between the grids; prior calculations consider only the dipole limit, in which the distance between grids is allowed to go to zero while the product of distance and amplitude of charge density on either grid remains finite.

The weakly nonlinear steady-state response to grid excitation is studied by a perturbation expansion of the Vlasov equation involving a smallness parameter proportional to the amplitude of grid excitation. The lowest order nonlinear response, which consists of components of zero frequency and twice the applied frequency, is considered. The problem is rendered tractable by approximating the potential in the linearized theory, which appears quadratically in the driving term of the equations determining nonlinear quantities of lowest order, by an exponentially damped term. This term is the dominant component of the linear response over a large range of the spatial variable. The method

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* Excitation of these waves was achieved by Wong, D'Angelo, and Motley.[3]

developed for evaluation of integrals in the linearized theory is applied with appropriate modifications to the Fourier inversion integrals which appear in the nonlinear theory.

1.2 Vlasov Equation

As a plasma becomes hotter and more rarefied, close interactions between particles become less important relative to long range collective interactions. In the limit in which close interactions may be neglected the Vlasov or correlationless kinetic equation is obtained. When the interactions between particles are Coulomb interactions, the Vlasov equation is given by the set[1]

$$\left(\frac{\partial}{\partial t}+\underline{v}\cdot\frac{\partial}{\partial \underline{x}}\right)F_\alpha(\underline{x},\underline{v},t)-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \underline{x}}\Phi(\underline{x},t)\cdot\frac{\partial}{\partial \underline{v}}F_\alpha(\underline{x},\underline{v},t)=0 \qquad (1.2.1)$$

$$-\frac{\partial}{\partial \underline{x}}\cdot\frac{\partial}{\partial \underline{x}}\Phi(\underline{x},t)=\frac{\rho_e(\underline{x},t)}{\epsilon_0}+\sum_\alpha \frac{q_\alpha n_{o\alpha}}{\epsilon_0}\int F_\alpha(\underline{x},\underline{v},t)\,d^3\underline{v}\ . \qquad (1.2.2)$$

For the two-species plasma considered, $\alpha = i, e,$ for ions and electrons, respectively; $n_{o\alpha}$ is the mean number density of species α (meters^{-3}) . The function $F_\alpha(\underline{x},\underline{v},t)$ is the one-body distribution function of species α . The quantity $n_{o\alpha}F_\alpha(\underline{x},\underline{v},t)d^3\underline{x}\ d^3\underline{v}$ is the mean number of particles of species α in the six-dimensional phase space volume element, $d^3\underline{x}\ d^3\underline{v}$, at $\underline{x},\underline{v}$ at time t . The external charge density which is the source of the external electric field is $\rho_e(\underline{x},t)$. The quantity $\Phi(\underline{x},t)$ is the Vlasov or self-consistent potential. Its sources are the external charge density and the plasma particles.

In order to linearize the Vlasov equation we resolve the species distribution functions into two terms,

$$F_\alpha(\underline{x},\underline{v},t) = f_{o\alpha}(\underline{v}) + f_\alpha(\underline{x},\underline{v},t) \ , \tag{1.2.3}$$

in which $f_{o\alpha}(\underline{v})$ describes a uniform, time-independent state of the plasma and $f_\alpha(\underline{x},\underline{v},t)$ is a small perturbation thereto. The velocity integral of $f_{o\alpha}(\underline{v})$ is normalized to unity. Since there is strict charge neutrality in the unperturbed state, the species charges and mean number densities satisfy the relation

$$\sum_\alpha q_\alpha n_{o\alpha} = 0 . \tag{1.2.4}$$

The Vlasov potential is likewise expanded into two terms,

$$\Phi(\underline{x},t) = 0 + \phi(\underline{x},t) \ . \tag{1.2.5}$$

In the unperturbed state described above, Φ is a constant which we set equal to zero. Since $f_\alpha(\underline{x},\underline{v},t)$ is small relative to $f_{o\alpha}(\underline{v})$, the term in the Vlasov equation involving the product ϕf_α may be neglected, giving for the linearized Vlasov equation the set

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}}\right) f_\alpha(\underline{x},\underline{v},t) - \frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \underline{x}}\phi(\underline{x},t)\cdot\frac{\partial}{\partial \underline{v}} f_{o\alpha}(\underline{v}) = 0 \tag{1.2.6}$$

$$-\frac{\partial}{\partial \underline{x}}\cdot\frac{\partial}{\partial \underline{x}}\phi(\underline{x},t)=\frac{\rho_e(\underline{x},t)}{\epsilon_0}+\sum_\alpha \frac{q_\alpha n_{0\alpha}}{\epsilon_0}\int f_\alpha(\underline{x},\underline{v},t)d^3\underline{v}\ . \tag{1.2.7}$$

1.3 Laplace-Fourier Transforms in Linearized Theory

In the remaining two sections of this chapter we develop a widely used method for the treatment of the linearized Vlasov equation. In the manner of Landau[2] the set of equations is Fourier transformed in space and Laplace transformed in time. Functions of a complex variable defined by velocity integrals which result are analytically continued throughout the plane of their argument. The Laplace-Fourier transform of the perturbation potential in the plasma is expressed in terms of the transform of the external charge density and of the dielectric functions which embody the response of the plasma in the linearized theory. The Fourier inversion integrals for the steady-state potential are exhibited.

The Fourier transform pair for the potential is

$$\phi(\underline{k},t)=\int d^3\underline{x}\, e^{-i\underline{k}\cdot\underline{x}}\,\phi(\underline{x},t) \tag{1.3.1}$$

$$\phi(\underline{x},t)=\int\frac{d^3\underline{k}}{(2\pi)^3}\,e^{i\underline{k}\cdot\underline{x}}\,\phi(\underline{k},t)\ . \tag{1.3.2}$$

A similar pair exists for $f_\alpha(\underline{x},\underline{v},t)$ and $f_\alpha(\underline{k},\underline{v},t)$. Taking the Fourier transform of the linearized Vlasov equation and assuming that perturbed quantities go to zero as $|\underline{x}|$ approaches infinity, we obtain

$$\left(\frac{\partial}{\partial t}+i\underline{k}\cdot\underline{v}\right)f_\alpha(\underline{k},\underline{v},t)-\frac{q_\alpha}{m_\alpha}\phi(\underline{k},t)\,i\underline{k}\cdot\frac{\partial}{\partial\underline{v}}f_{o\alpha}(\underline{v})=0 \tag{1.3.3}$$

$$k^2\phi(\underline{k},t)=\frac{\rho_e(\underline{k},t)}{\epsilon_o}+\sum_\alpha\frac{q_\alpha n_{o\alpha}}{\epsilon_o}\int f_\alpha(\underline{k},\underline{v},t)\,d^3\underline{v}\ . \tag{1.3.4}$$

The Laplace transform pair for the potential is

$$\phi(\underline{k},\omega)=\int_0^\infty dt\,e^{i\omega t}\,\phi(\underline{k},t) \tag{1.3.5}$$

$$\phi(\underline{k},t)=\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{d\omega}{2\pi}\,e^{-i\omega t}\,\phi(\underline{k},\omega)\ . \tag{1.3.6}$$

A similar pair exists for $f_\alpha(\underline{k},\underline{v},t)$ and $f_\alpha(\underline{k},\underline{v},\omega)$. The transforms exist for $\gamma = \mathrm{Im}\{\omega\} > \gamma_o$, where $\mathrm{Im}\{\omega\} = \gamma_o$ lies above all singularities of the transforms. The inversion contour specified above is denoted by L . The Laplace-Fourier transformed Vlasov equation is the set

$$i(\underline{k}\cdot\underline{v}-\omega)f_\alpha(\underline{k},\underline{v},\omega)-\frac{q_\alpha}{m_\alpha}\phi(\underline{k},\omega)\,i\underline{k}\cdot\frac{\partial}{\partial\underline{v}}f_{o\alpha}(\underline{v})=f_\alpha(\underline{k},\underline{v},t=0) \tag{1.3.7}$$

$$k^2\phi(\underline{k},\omega)=\frac{\rho_e(\underline{k},\omega)}{\epsilon_o}+\sum_\alpha\frac{q_\alpha n_{o\alpha}}{\epsilon_o}\int f_\alpha(\underline{k},\underline{v},\omega)\,d^3\underline{v}\ . \tag{1.3.8}$$

The quantity $f_\alpha(\underline{k},\underline{v},t = 0)$ is the Fourier transform of the initial perturbation to the distribution function of species α. The unperturbed species distribution functions which we consider are Maxwellian with no drift velocity,

$$f_{o\alpha}(\underline{v}) = \frac{e^{-v^2/a_\alpha^2}}{(\sqrt{\pi}\, a_\alpha)^3} \quad , \tag{1.3.9}$$

where $v^2 = v_x{}^2 + v_y{}^2 + v_z{}^2$ and a_α is the thermal speed of species α. We note that the velocity integral of this distribution function is properly normalized to unity. A plasma whose unperturbed state is described by these distribution functions is stable: initial perturbations to the distribution functions are damped in time.[1] We may as well assume, therefore, that there are no initial perturbations to the distribution functions; hence, we put $f_\alpha(\underline{k},\underline{v},t = 0)$ equal to zero.

We consider excitation of the plasma by an idealized pair of grids which produces an external charge density with a spatial variation only in the x-direction: $\rho_e(\underline{x},t) = \rho_e(x,t)$. Taking the Laplace-Fourier transform of this charge density in accordance with Equations (1) and (5), we have

$$\rho_e(\underline{k},\omega) = \rho_e(k_x,\omega)\left[2\pi\,\delta(k_y)\right]\left[2\pi\,\delta(k_z)\right] \quad . \tag{1.3.10}$$

The external field produced by this charge density affects only the x-component of the velocity of a particle. Any Fourier component of a perturbation in the plasma which is not strictly in the x-direction is uncoupled from the external excitation and is not considered here. Therefore $f_\alpha(\underline{k},\underline{v},\omega)$ and $\phi(\underline{k},\omega)$ have the same functional dependence on $\underline{k}$ as $\rho_e(\underline{k},\omega)$; furthermore, we have $\underline{k}\cdot\underline{v} = k_x v_x$ and $\underline{k}\cdot(\partial/\partial\underline{v}) = k_x(\partial/\partial v_x)$. We integrate over velocity components perpendicular to the x-direction, which appear only parametrically. We obtain for the one-dimensional Vlasov equation the set

$$ik_x(v_x-\omega/k_x)f_\alpha(k_x,v_x,\omega)-\frac{q_\alpha}{m_\alpha}\phi(k_x,\omega)ik_x\frac{\partial}{\partial v_x}f_{o\alpha}(v_x)=0 \qquad (1.3.11)$$

$$k_x^2\phi(k_x,\omega)=\frac{\rho_e(k_x,\omega)}{\epsilon_o}+\sum_\alpha\frac{q_\alpha n_{o\alpha}}{\epsilon_o}\int_{-\infty}^{\infty}f_\alpha(k_x,v_x,\omega)dv_x\,, \qquad (1.3.12)$$

in which $f_{o\alpha}(v_x)$ is the one-dimensional Maxwellian distribution function, $e^{-v_x^2/a_\alpha^2}/(\sqrt{\pi}\,a_\alpha)$. Henceforth we suppress the x-subscripts in Equations (11) and (12).

The Laplace-Fourier transform of the potential in the plasma is obtained from Equations (11) and (12) by eliminating $f_\alpha(k,v,\omega)$. We have

$$\phi(k,\omega)=\frac{\rho_e(k,\omega)}{\epsilon_o k^2\left[1-\sum_\alpha\frac{\omega_{p\alpha}^2}{k^2}\int_{-\infty}^{\infty}\frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega/k)}dv\right]} \qquad (1.3.13)$$

in which $\omega_{p\alpha} = (q_\alpha^2 n_{o\alpha}/\epsilon_o m_\alpha)^{1/2}$ is the plasma frequency of species α .

We shall need for the inversion of this transform the analytic continuation of the functions defined by the velocity integrals of Equation (13) into the entire plane of the argument $\zeta_\alpha = \omega/ka_\alpha$. With this analytic continuation the quantity in square brackets is the dielectric function of the plasma and is denoted by $K(k,\omega)$.

The Laplace-Fourier inversion of $\phi(k,\omega)$ is

$$\phi(x,t) = \int_L \frac{d\omega}{2\pi} e^{-i\omega t} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \frac{\rho_e(k,\omega)}{\epsilon_0 k^2 K(k,\omega)} \quad . \tag{1.3.14}$$

It was noted above that an unperturbed plasma described by Maxwellian distribution functions with no drift velocity is stable against the growth in time of initial perturbations. This conclusion is derived from the fact that for real k there are no zeros of the dielectric function in the upper half ω plane or on the real ω axis. In the present problem, therefore, a steady state exists in which the temporal behavior of the plasma is determined by the poles of $\rho_e(k,\omega)$ in the ω plane. We assume that the external charge density produced by the pair of grids is of the form

$$\rho_e(x,t) = \rho_e(x) \cos \omega_0 t \qquad (t \geq 0) . \tag{1.3.15}$$

Accordingly the steady-state potential in the plasma is given by

$$\phi(x,t) = \frac{e^{-i\omega_0 t}}{2} \left[\frac{1}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho_e(k) e^{ikx}}{k^2 K(k,\omega_0)} dk \right] + \frac{e^{i\omega_0 t}}{2} \left[\frac{1}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho_e(k) e^{ikx}}{k^2 K(k,-\omega_0)} dk \right] . \tag{1.3.16}$$

The second term is the complex conjugate of the first. The potential is also given by twice the real part of the first term.

1.4 Analytic Continuation of Functions Defined by Velocity Integrals

The analytic continuation of the function of the complex variable $\zeta_\alpha = \omega/ka_\alpha$ defined by the velocity integral

$$\int_{-\infty}^{\infty} \frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega/k)} dv = \frac{1}{a_\alpha^2} \frac{d}{d\zeta_\alpha} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{\pi}(s-\zeta_\alpha)} ds \tag{1.4.1}$$

is determined in the manner of Landau.[2] We consider the function $Z(\zeta_\alpha)$, which is defined by the integral

$$Z(\zeta_\alpha) = \int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{\pi}(s-\zeta_\alpha)} ds \tag{1.4.2}$$

when ω and k are on the primitive inversion contours, L and the real axis in the k plane, respectively. If k is positive the integral defines a function of ζ_α which is regular and analytic in the upper half plane. If k is negative the integral defines a function which is regular and analytic in the lower half plane. Two functions of ζ_α are thus defined; the real axis of the ζ_α plane is a branch-cut of the functions.

These matters are clarified by considering a simpler integral along the real axis of the s plane between $s = -c$ and $s = c$,

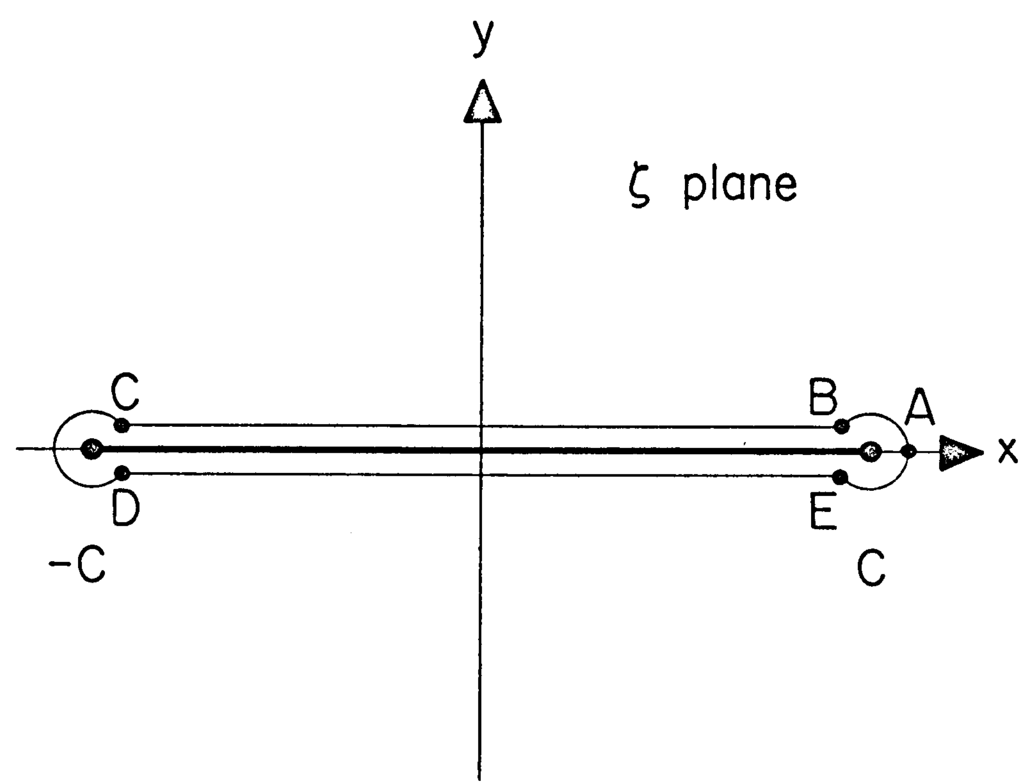
$$\Lambda(\zeta) = \int_{-c}^{c} \frac{ds}{(s-\zeta)} = \log\left(\frac{\zeta - c}{\zeta + c}\right) , \tag{1.4.3}$$

which defines a regular analytic function of the complex variable ζ provided that ζ does not lie on the path of integration. The function $\Lambda(\zeta)$ has logarithmic branch points at $\zeta = \pm c$, which must be joined by a branch-cut along the real axis of the ζ plane. See Figure 1. By considering the closed contour surrounding the branch-cut shown in the figure, we conclude that

$$\Lambda_+(x) - \Lambda_-(x) = \begin{cases} 2\pi i & (|x| < c) \\ 0 & (|x| > c) \end{cases} \tag{1.4.4}$$

where $\Lambda_+(x)$ is the limiting value of $\Lambda(\zeta)$ when ζ approaches the branch-cut from above and $\Lambda_-(x)$ is the limiting value when ζ approaches the branch-cut from below. The integral of Equation (2) may be considered the limit, as $|c| \to \infty$, of a corresponding integral over finite limits $\pm c$. The presence of the factor e^{-s^2} does not affect the character of the branch-cut at the limits of integration.

The analytic continuation of the functions defined by Equation (2) is obtained by deforming the contour of integration to avoid the singularity at $s = \zeta_\alpha$. Explicitly the two functions, as analytically continued, are



BRANCH-CUT OF THE FUNCTION $\Lambda(\zeta)$

FIGURE 1.

$$Z_{\pm}(\zeta_\alpha) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{\pi}(s-\zeta_\alpha)} ds & \left(\mathrm{Im}\{\zeta_\alpha\} \gtrless 0\right) \quad (1.4.5) \\ \displaystyle P\int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{\pi}(s-\zeta_\alpha)} ds \pm \sqrt{\pi}\, i e^{-\zeta_\alpha^2} & \left(\mathrm{Im}\{\zeta_\alpha\} = 0\right) \quad (1.4.6) \\ \displaystyle\int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{\pi}(s-\zeta_\alpha)} ds \pm 2\sqrt{\pi}\, i e^{-\zeta_\alpha^2} & \left(\mathrm{Im}\{\zeta_\alpha\} \lessgtr 0\right). \quad (1.4.7) \end{cases}$$

The plus function, $Z_+(\zeta_\alpha)$, is the plasma dispersion function treated by Fried and Conte.[5]

Corresponding to the plus and minus functions of Equations (5) - (7) are the two dielectric functions

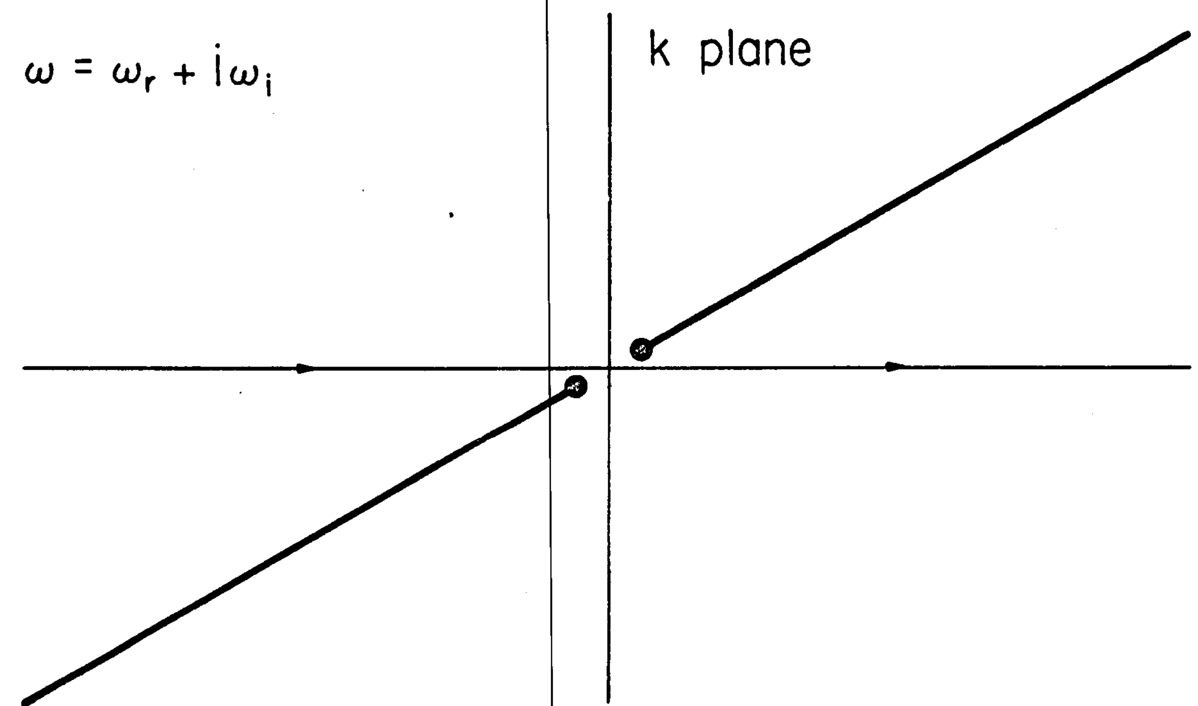
$$K_{\pm}(k,\omega) = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2 a_\alpha^2} Z_{\pm}'\left(\frac{\omega}{k a_\alpha}\right) . \tag{1.4.8}$$

Accordingly Equation (1.3.16) may be rewritten by bisecting the primitive contour of integration into positive and negative parts and by indicating the appropriate dielectric function on each part. The result is

$$\phi(x,t) = \frac{e^{-i\omega_0 t}}{2}\left[\frac{1}{2\pi\epsilon_0}\int_{-\infty}^{0} \frac{\rho_e(k)e^{ikx}}{k^2 K_-(k,\omega_0)} dk + \frac{1}{2\pi\epsilon_0}\int_{0}^{\infty} \frac{\rho_e(k)e^{ikx}}{k^2 K_+(k,\omega_0)} dk\right] + \text{c.c.} \tag{1.4.9}$$

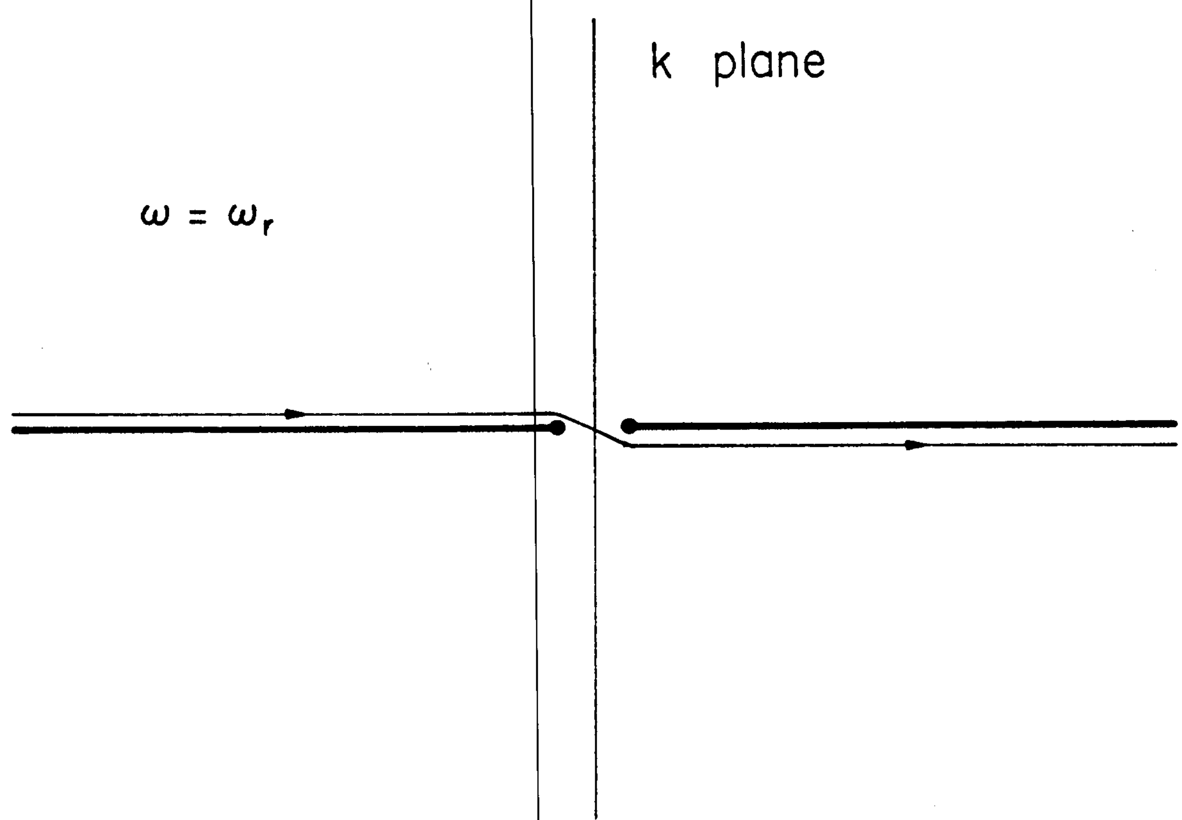
An infinite integral involving K without subscripts will be used frequently below to denote the situation described by Equation (9).

We can determine the relation between the integration contour in the k plane and the mapping onto the k plane of the branch-cut in the ζ_α plane by considering the velocity integrals in the dielectric functions to have finite limits of integration. For a particular value of ω on the primitive Laplace inversion contour L , viz. $\omega = \omega_r + i\omega_i$, which we choose for definiteness to have a positive real part, the mapping of the branch-cut onto the k plane, $k = (\omega_r/a_\alpha\zeta_\alpha) + i(\omega_i/a_\alpha\zeta_\alpha)$, is shown in Figure 2. As the limits of integration approach $\pm\infty$ the branch points approach the origin on opposite sides of the contour of integration. As ω approaches ω_o the mapping of the branch-cut coincides with the real axis of the k plane; the orientation of the contour of integration relative to the branch-cut is shown in Figure 3.



MAPPING OF BRANCH-CUT ONTO k PLANE FOR COMPLEX ω.

FIGURE 2.



MAPPING OF BRANCH-CUT ONTO k PLANE FOR REAL ω.

FIGURE 3.

II. RESPONSE TO LOW FREQUENCY GRID EXCITATION IN LINEARIZED THEORY

The character of grid excitation considered is now specified and the numerical evaluation of the Fourier inversion integral of Equation (1.4.9) is carried out. We consider the driving frequency, ω_o , to be below the ion plasma frequency, ω_{pi} , or, at most, comparable in magnitude with it.* Both electron and ion motions are excited in this frequency range. The response near the grids in these circumstances is often referred to as an ion acoustic wave.[3,4] We consider two cases. In one case there is a finite separation between the two idealized parallel plane grids. In the other case the dipole limit for the pair of grids is assumed.

In Section 1 the fundamental integrals for the two cases are formulated as branch-cut integrals along the positive real axis in the k plane. Section 2 describes the transformation of variable of integration from k to $\zeta = \omega_o/ka_i$ and the separation of the integrand into two parts, one principally responsible for the ion wave which is dominant near the grid, and the other for the electron wave which is dominant at large distances from the grid. In Section 3 the integration contour of the resulting ion integral is deformed onto the path of steepest descents through an appropriate saddle point; except at very small distances from the grid the deformation results in a residue contribution which is much larger than the integral along the path of

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* To simplify the calculation we later make the assumption that $\omega_o/\omega_{pi} << 1$, which permits a simplification that leads to negligible errors [4] for $\omega_o/\omega_{pi} \lesssim .3$.

steepest descents. Section 4 contains an analysis of the advantages of the present method over that of Gould.[4] This provides guidance in choosing a deformed contour for the electron integral, which is described in Section 5. Section 6 describes the computation of the integrals and presents the numerical results.

2.1 Fundamental Integrals for Grid Excitation: Dipole Limit and Finite Separation Cases

The spatial behavior of the external charge density produced by the idealized pair of grids is $\rho_e(x) = \sigma_o[\delta(x-x_o/2) - \delta(x+x_o/2)]$, in which σ_o is the amplitude of the surface charge density on either grid and x_o is the separation between the grids. In the dipole limit, $x_o \to 0,\ \sigma_o x_o = \text{constant}$, it becomes $\rho_e^{(d)}(x) = -\sigma_o x_o (d/dx)\,\delta(x)$. The corresponding Fourier transforms are $\rho_e(k) = \sigma_o(e^{-ikx_o/2} - e^{ikx_o/2})$ and $\rho_e^{(d)}(k) = -i\sigma_o x_o k$, respectively.

It is convenient to express Equation (1.4.9) as $\phi(x,t) = [e^{-i\omega_o t}/2]\ \phi(x) + \text{c.c.}$, in which we have

$$\phi(x) = -i\frac{\sigma_o x_o}{2\pi\epsilon_o}\left\{\int_{-\infty}^{0}\frac{e^{ikx}}{kK_-(k,\omega_o)}\left[\frac{\sin(kx_o/2)}{(kx_o/2)}\right]dk + \int_{0}^{\infty}\frac{e^{ikx}}{kK_+(k,\omega_o)}\left[\frac{\sin(kx_o/2)}{(kx_o/2)}\right]dk\right\}. \tag{2.1.1}$$

We are considering a two-component plasma, for which

$$K_\pm(k,\omega_o) = 1 - \frac{\omega_{pi}^2}{k^2 a_i^2} Z'_\pm\left(\frac{\omega_o}{ka_i}\right) - \frac{\omega_{pe}^2}{k^2 a_e^2} Z'_\pm\left(\frac{\omega_o}{ka_i}\right). \tag{2.1.2}$$

The only singularity of the integrand of the first integral in

Equation (2) in the upper half k plane is a simple pole at $k \simeq 2^{1/2}\, i[(\omega_{pe}/a_e)^2 + (\omega_{pi}/a_i)^2]^{1/2} = ik_D$. We consider positive values of x . (The integral is an odd function of x .) For points outside the grids, $x > x_o/2$, we deform the contour of integration of the first integral in Equation (1) in the upper half plane so that it runs from $+\infty$ to 0 , as shown in Figure 4.* The residue contribution from the pole at $k \simeq ik_D$ is much more heavily damped in space than the branch-cut integral[4] and is therefore neglected. The restriction to values of x outside the grids assures that the integrand vanishes at infinity when the folding is carried out as described here.** We now have

$$\phi(x) = -i\frac{\sigma_o x_o}{2\pi\epsilon_o}\int_0^\infty \frac{e^{ikx}}{k}\left[\frac{\sin(kx_o/2)}{(kx_o/2)}\right]\left[\frac{1}{K_+(k,\omega_o)} - \frac{1}{K_-(k,\omega_o)}\right]dk\,. \qquad (2.1.3)$$

In the dipole limit this becomes

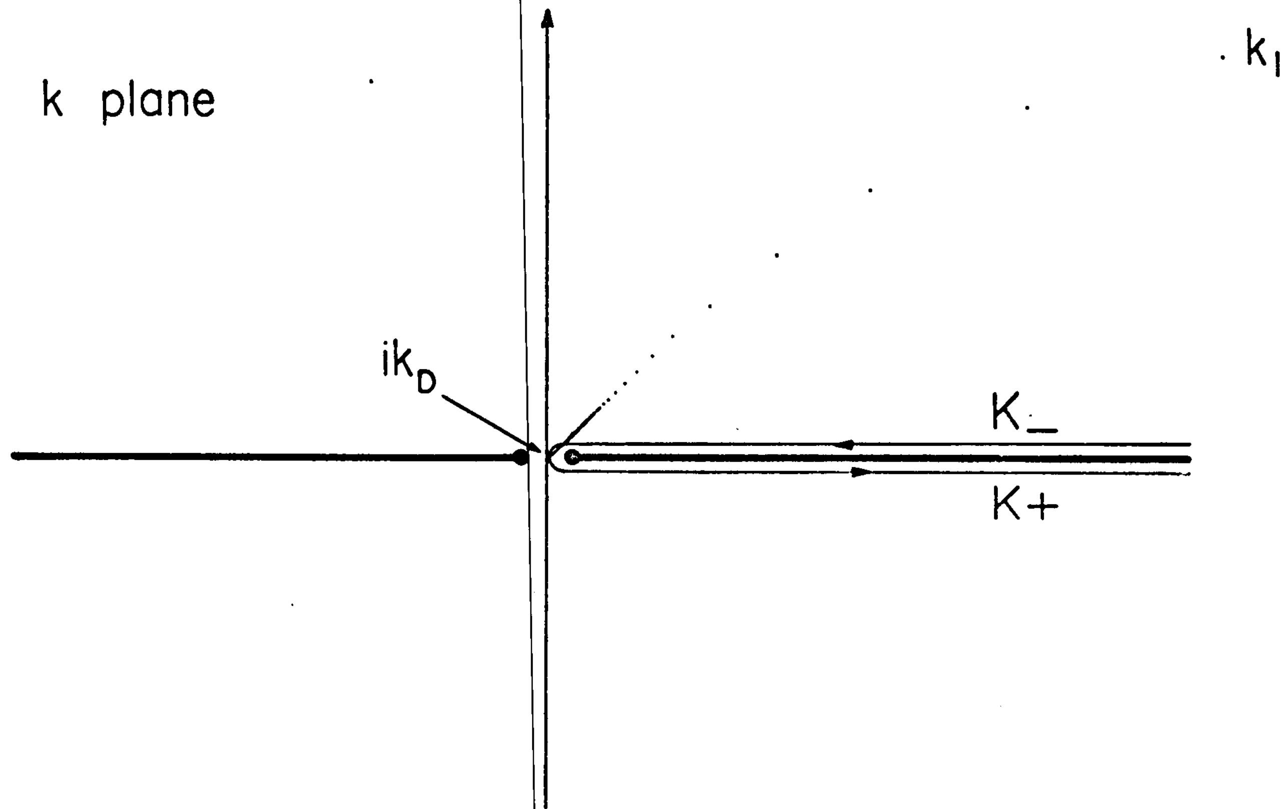
$$\phi_1(x) = -i\frac{\sigma_o x_o}{2\pi\epsilon_o}\int_0^\infty \frac{e^{ikx}}{k}\left[\frac{1}{K_+(k,\omega_o)} - \frac{1}{K_-(k,\omega_o)}\right]dk\,. \qquad (2.1.4)$$

There is a simple pole of the integrand of Equation (1) at $k = 0$. This contributes to the potential a constant, different on either side

---

* In Figure 4 are shown also the zeros of K_+ . The zero of K_+ indicated by k_1 is particularly important in that, at distances neither too close to, nor too far from, the grid, the response is closely approximated by an exponentially damped wave with this complex wave number.[4]

** Determination of the potential for points $0 \leq x < x_o/2$ is discussed below.



FOLDED CONTOUR OF INTEGRATION IN k PLANE FOR INTEGRAL IN LINEARIZED THEORY.

FIGURE 4.

of the grid, which may be neglected. It is shown below [see Equation (2.2.6)] that the difference $(K_- - K_+)$ approaches zero exponentially as k approaches zero. The integrals $\phi(x)$ and $\phi_1(x)$ of Equations (3) and (4), therefore, do not exhibit a contribution from this pole.

The integral for $\phi(x)$ can be expressed as the difference

$$\phi(x) = x_o^{-1}\left[\phi_2(x + x_o/2) - \phi_2(x - x_o/2)\right] \tag{2.1.5}$$

where

$$\phi_2(x) = -\frac{\sigma_o x_o}{2\pi\epsilon_o}\int_0^\infty \frac{e^{ikx}}{k^2}\left[\frac{1}{K_+(k,\omega_o)} - \frac{1}{K_-(k,\omega_o)}\right]dk \tag{2.1.6}$$

is the fundamental integral for finite grid separation.

For points inside the grid pair, $0 \leq x < x_o/2$, $\sin(kx_o/2)$ must be replaced by its exponential representation before the primitive contour is folded. The contour of the fundamental integral with argument $(x + x_o/2)$ is folded in the usual manner; the positive half of the primitive contour of the fundamental integral with argument $(x - x_o/2)$ is folded in the lower half plane onto the negative half of the real k axis. Introducing the transformation $k \to -k$ and using the identity $K'_\pm(-k,\omega_o) = K_\mp(k,\omega_o)$ we conclude that the potential $\phi(x)$ may be obtained for positive x inside and outside the grid pair by the following extension of Equation (5):

$$\phi(x) = x_o^{-1}\left[\phi_2(x + x_o/2) - \phi_2(|x - x_o/2|)\right] \ . \tag{2.1.7}$$

In the following sections the development is carried out with reference to $\phi_1(x)$. Numerical calculations of both $\phi_1(x)$ and $\phi_2(x)$ are performed.

2.2 Transformation of Integration Variable and Separation Into Two Integrals

Except for the inclusion of finite grid separation, the formulation follows Gould[4] up to this point. We now diverge from his method; a comparison of the two methods will be made below.

In order to proceed further we give to x a small positive imaginary part, i.e., $x = |x|e^{i\delta}$, with $\delta > 0$. This step is necessary to the transformation of variable of integration now to be introduced. When deformed contours of integration are introduced below, δ may be set equal to zero.

We introduce the following dimensionless variables, which are appropriate to the excitation at the frequencies which we consider:

$$z \equiv \frac{\omega_0}{a_i} x \,;\; f \equiv \frac{\omega_0}{\omega_{pi}} \,;\; \Phi_1 \equiv -\frac{\epsilon_0}{\sigma_0 x_0}\phi_1 \,;\; \tilde{T} \equiv \frac{T_i}{T_e} \,. \tag{2.2.1}$$

The species kinetic temperatures are $T_\alpha = m_\alpha a_\alpha^2/2K$, where K is the Boltzmann constant. We introduce also the mass ratio, $\mu = m_e/m_i$. Introducing these variables and transforming the variable of integration from k to $\zeta \equiv \omega_0/ka_i$, we obtain

$$\frac{\Phi_1}{f^2} = \frac{i}{2\pi}\int_0^\infty \frac{e^{iz/\zeta}}{\zeta}\left[\frac{1}{f^2K_+(\zeta,f^2)} + \frac{-1}{f^2K_-(\zeta,f^2)}\right] d\zeta \tag{2.2.2}$$

in which

$$f^2 K_{\pm}(\zeta, f^2) = f^2 - \zeta^4 \left[Z_{\pm}'(\zeta) + \tilde{T} \, Z_{\pm}'(\mu^{1/2} \tilde{T} \zeta) \right] . \tag{2.2.3}$$

We observe that the integral consists of the sum of two integrals on opposite sides of the branch-cut in the ζ plane. See Figure 5.

The reason for giving x a small positive imaginary part is now apparent. The factor $e^{iz/\zeta}$ has an essential singularity at $\zeta = 0$. The imaginary part of x causes $e^{iz/\zeta}$ to be exponentially damped as $\zeta \to 0$, thus assuring the existence of the integral. If ζ approaches the origin coming from the lower half plane as in the deformed contours to be chosen, $e^{iz/\zeta}$ is exponentially damped as $|\zeta| \to 0$ for real z, and δ may be set equal to zero.

The functions $Z_{\pm}(\zeta)$ have integral representations[5]

$$Z_{\pm}(\zeta) = e^{-\zeta^2} \left[\pm i\sqrt{\pi} - 2\int_0^{\zeta} e^{t^2} dt \right] \tag{2.2.4}$$

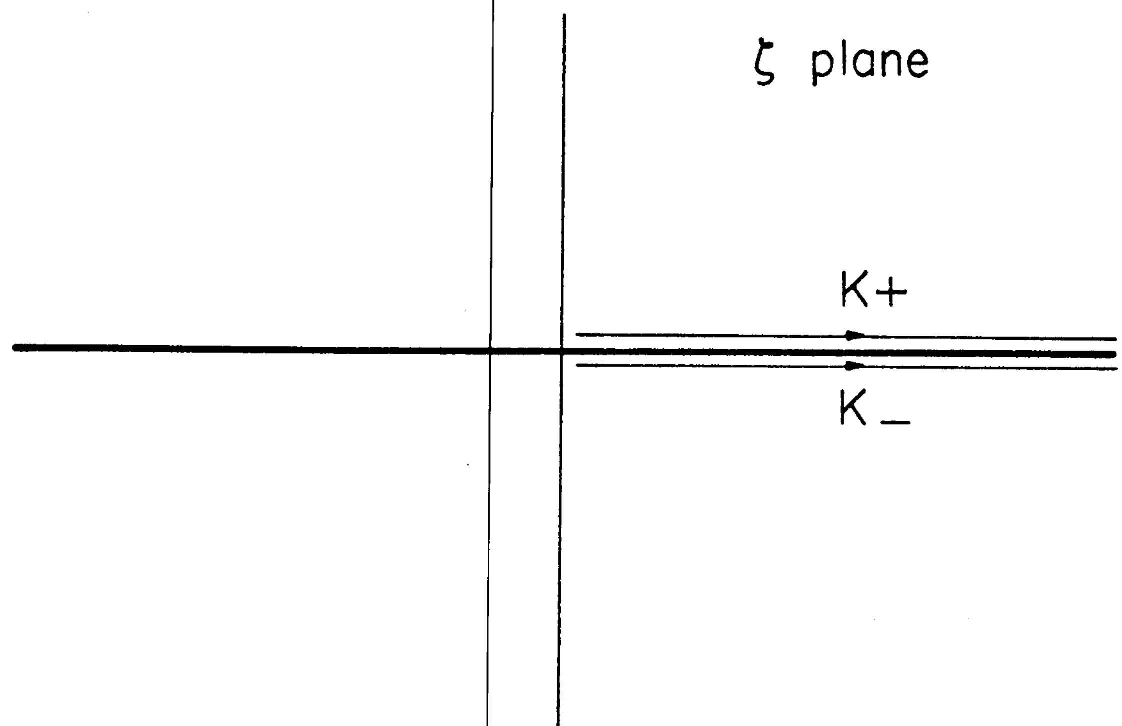
which are valid throughout the finite ζ plane. These representations, or Equations (1.4.5) - (1.4.7), give

$$Z_{+}(\zeta) - Z_{-}(\zeta) = 2\sqrt{\pi}\, i\, e^{-\zeta^2} \tag{2.2.5}$$

from which there follows

$$K_{-}(\zeta, f^2) - K_{+}(\zeta, f^2) = -\frac{4\sqrt{\pi}\, i}{f^2} \zeta^3 \left(e^{-\zeta^2} + \mu^{1/2} \tilde{T}^{3/2} e^{-\tilde{T}\mu\zeta^2} \right) . \tag{2.2.6}$$

With this relation we obtain



INITIAL CONTOUR OF FUNDAMENTAL INTEGRAL IN ζ PLANE.

FIGURE 5.

$$\frac{\Phi_1}{f^2}=\frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{iz/\zeta-\zeta^2}}{\zeta^{-2}[f^2K_+(\zeta,f^2)][f^2K_-(\zeta,f^2)]}d\zeta+\mu^{1/2}\frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{iz/\zeta-\mu\zeta^2}}{\zeta^{-2}[f^2K_+(\zeta,f^2)][f^2K_-(\zeta,f^2)]}d\zeta \qquad (2.2.7)$$

in which we have taken the equal temperature case, $\tilde{T} = 1$, to which our attention will be confined. The difference $(K_- - K_+)$ approaches zero exponentially as $k \to 0$ $(\zeta \to \infty)$.

The exponential attenuation of $e^{iz/\zeta}$ as $|\zeta| \to 0$, assured now by $\delta > 0$ and later by the approach of the deformed contour to $\zeta = 0$ coming from the lower half plane, permits us to consider the case $f^2 << 1$, to which we shall limit our attention. We denote $\lim\limits_{f^2\to 0}[f^2K_\pm(\zeta,f^2)]$ by

$$f^2K_\pm(\zeta) = -\zeta^2\left[Z'_\pm(\zeta) + Z'_\pm(\mu^{1/2}\zeta)\right] \qquad (2.2.8)$$

and observe that in this limit Φ_1/f^2 is independent of f^2 . The exponential attenuation of $e^{iz/\zeta}$ as $|\zeta| \to 0$ discussed above takes care of the difficulty caused by the vanishing of $f^2K_\pm(0)$.

The factors $e^{-\zeta^2}$ and $e^{-\mu\zeta^2}$ place upper limits on the ranges of $|\zeta|$ which contribute substantially to the two integrals, namely $|\zeta| \lesssim O(1)$ and $|\zeta| \lesssim O(\mu^{-1/2})$, respectively. Recalling the definition of ζ and the relation $a_i/a_e = \mu^{1/2}$ (for $\tilde{T} = 1$) , we see that the first (second) integral has substantial contributions from values of the phase velocity ω_o/k which are not greater in order of magnitude than the ion (electron) thermal velocity. It is appropriate, therefore, to refer to the first (second) integral as the ion (electron) integral.

2.3 Path of Steepest Descents for Ion Integral

We now consider a deformed contour in the ζ plane which coincides with the path of steepest descents passing through an appropriate saddle point of the exponent in the function $e^{iz/\zeta-\zeta^2}$. This contour will be used for the numerical evaluation of the ion integral. In order to avoid confusion concerning deformation of the upper contour of Figure 5 through the branch-cut, we shall first deform the contour of the total integral, Equation (2.2.2), onto the path of steepest descents for the ion integral. Subsequently we shall deform the contour of integration of the electron integral further to a contour appropriate to it.

An integral whose integrand determines the path of steepest descents sought is

$$I = \int_0^{\infty} \chi(\zeta)\, e^{iz/\zeta - \zeta^2}\, d\zeta. \tag{2.3.1}$$

Making a transformation of the variable of integration to $w = z^{-1/3}\zeta$, and considering integration over an appropriate deformed contour C, there results[6]

$$I = z^{1/3} \int_C \chi(z^{1/3} w)\, e^{z^{2/3}\varphi(w)}\, dw \tag{2.3.2}$$

in which $\phi(w) = (i/w) - w^2$. Determination of the saddle points by the condition $\phi'(w) = 0$ yields

$$w_0 = z^{-1/3} e^{-\pi i/6}, \quad w_1 = z^{-1/3} e^{\pi i/2}, \quad w_2 = z^{-1/3} e^{-7\pi i/6}. \tag{2.3.3}$$

The proper saddle point for this integral is w_0 ; $\phi(w_0) = (3/2^{2/3})\, e^{2\pi i/3}$. The path of steepest descents is determined by the conformal transformation

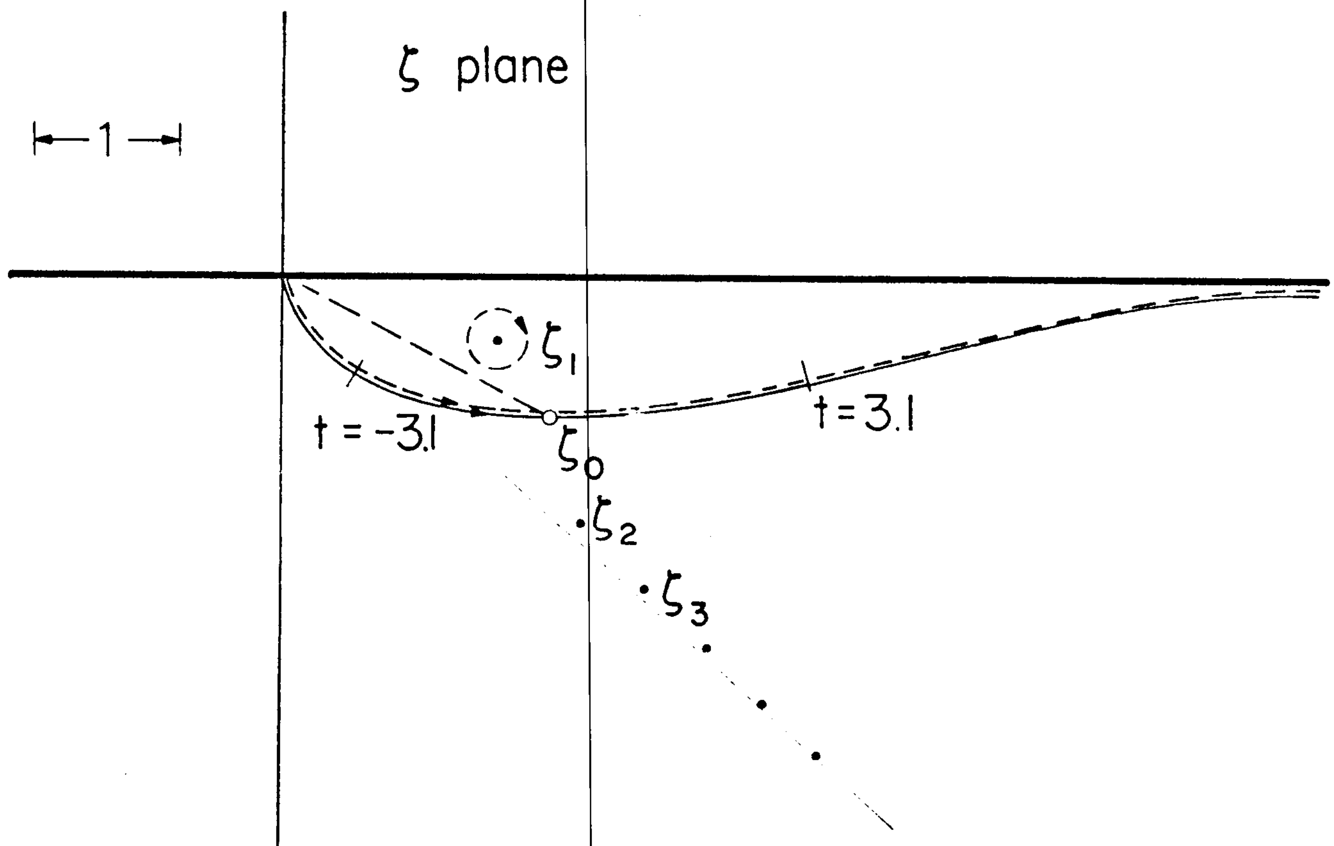
$$t^2 = z^{2/3}\left[\phi(w_0) - \phi(w)\right] \tag{2.3.4}$$

in which t is real and varies between $-\infty$ and $+\infty$. The integral becomes

$$I = z^{1/3} e^{z^{2/3}\phi(w_0)} \int_{-\infty}^{\infty} e^{-t^2} \chi\left[z^{1/3} w(t)\right] \frac{dw}{dt}\, dt + Q \tag{2.3.5}$$

in which Q denotes the contribution to I from the residues of any poles swept in the ζ plane between the positive real axis and the mapping of the path of steepest descents onto the ζ plane. The numerical evaluation of the ion integral is carried out over a range of t sufficient to ensure negligible error. The presence of the factor e^{-t^2} permits limitation of the range of numerical integration to $-3.1 \le t \le 3.1$ for the achievement of satisfactory accuracy. The inversion of the conformal transformation, i.e., the determination of $w(t)$, is effected by an iterative technique described in Appendix A.

The character of the path of steepest descents in the ζ plane ($\zeta = z^{1/3}w$) is shown in Figure 6. The saddle point is located at



PATH OF STEEPEST DESCENTS FOR ION INTEGRAL.

FIGURE 6.

$\zeta_o = z^{1/3} w_o$. The depiction of the contour as a double solid-dashed line reflects the fact that the integral of Equation (2.2.2) is the sum of two integrals on opposite sides of the branch-cut in the ζ plane. The contour of integration of the integral containing K_+ , having crossed the branch-cut, lies on a Riemann sheet different from that on which $Z_+(\zeta)$ is defined as an integral over a real contour of integration; $Z_+'(\zeta)$ and $Z_+'(\mu^{1/2}\zeta)$ denote analytically continued functions for contours lying in the lower half ζ plane. Having deformed the contour of integration so that it approaches $\zeta = 0$ coming from the lower half plane, it is possible to set $\delta = 0$, as discussed above.

The first few zeros of $[f^2 K_+(\zeta)]$ are indicated in Figure 6. The transformation of variable of integration from k to ζ has cast the infinite set of zeros indicated in Figure 4 out toward large $|\zeta|$. The first zero, k_1 , which approximates closely the exponential damping over a considerable range of z , now appears closest to the origin, at $\zeta_1 \approx 1.45 - 0.60i$. As z increases, the path of steepest descents dips farther into the lower half ζ plane. At $z \approx 3.8$, the contour sweeps past ζ_1 and the contribution of the residue from the pole must be included for larger values of z . That contribution to Φ_1/f^2 is $[e^{iz/\zeta_1}/\{\zeta_1[f^2 K_+'(\zeta_1)]\}]$. As z increases, the path of steepest descents sweeps farther into the lower half ζ plane, but it does not reach the next pole, ζ_2 , in the range of z which we consider $(z \lesssim 60)$.

2.4 Advantages of Present Method

We now discuss the advantages of the method adopted to this point for the evaluation of Φ_1/f^2. The considerations introduced will provide guidance concerning the choice of a contour of integration for the electron integral, which involves additional complexities.

The method of Gould[4] for the evaluation of the integral of Equation (2.1.4) made use of the relation $K_-(k,\omega_o) = K_+(-k,\omega_o)$ and the fact that the real and imaginary parts of $Z_+'(\zeta)$ are, respectively, even and odd functions of the (real) argument ζ to give (for $\tilde{T} = 1$ and $f^2 \ll 1$)

$$\frac{\Phi_1}{f^2} = \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\frac{\eta}{Z_+'(1/\eta) + Z_+'(\mu^{1/2}/\eta)}\right] e^{i\eta z}\, d\eta \qquad (2.4.1)$$

in which $\eta \equiv ka_i/\omega_o$. This integral was evaluated numerically on the positive real η axis.

The appearance of a separate residue contribution which gives the dominant behavior of the response over a considerable range of z is one advantage of the present method over that just described. A second advantage is that the use of the path of steepest descents for the ion integral involves a small $[O(2\pi)]$ phase change of the integrand over the range of the contour required in the numerical integration. By contrast, the cut-off of the numerical integration of Equation (1) for large η is determined by the coefficient of $e^{i\eta z}$, which decreases slowly with increasing η. The securing of this advantage for the present method depends on choosing a deformed contour for the electron integral which involves a small phase change of the integrand over the range of the numerical integration. Such a contour is described below. A third advantage of the present method is the separation into two integrals with distinct ranges of variable of integrations, $|\zeta| \sim O(1)$ and $|\zeta| \sim O(\mu^{-1/2})$, which are responsible, respectively, for the ion and electron waves.* The integrands of the ion and electron integrals differ only in the presence of the factors $e^{-\zeta^2}$ and $\mu^{1/2}e^{-\mu\zeta^2}$, respectively. In the range $|\zeta| \lesssim O(1)$, the integrand of the electron integral is negligible compared to that of the ion integral.

2.5 Deformed Contour for Electron Integral

The path of steepest descents passing through the appropriate saddle point of the exponent in $e^{iz/\zeta-\mu\zeta^2}$ is obtainable by the method described in connection with the ion integral, with the replacements $z \rightarrow y \equiv \mu^{1/2}z$, $\zeta \rightarrow \xi \equiv \mu^{1/2}\zeta$. The form of integral considered now is

$$E = \int_0^\infty \psi(\zeta) e^{iz/\zeta - \mu\zeta^2} d\zeta \tag{2.5.1}$$

which may be written as

$$E = \mu^{-1/2} \int_0^\infty \psi(\mu^{-1/2}\xi) e^{iy/\xi - \xi^2} d\xi . \tag{2.5.2}$$

---

* See Figure 8. The electron integral makes no substantial contribution to the response for values of z less than about 20.

In accordance with our earlier procedure we define $w = y^{-1/3}\xi$. The integral is transformed into

$$E=\mu^{-1/2}y^{1/3}\int_C \psi(\mu^{-1/2}y^{1/3}w)e^{y^{2/3}\varphi(w)}dw \qquad (2.5.3)$$

where $\phi(w)$, w_o , and $\phi(w_o)$ are as before. The conformal transformation for the path of steepest descents is of the form of Equation (2.3.4), with $z \to y$. The path of steepest descents in the ζ plane is given by $\zeta = \mu^{-1/2}\xi = \mu^{-1/2}[y^{1/3}w(t)] = \mu^{-1/3}z^{1/3}w(t)$; $\zeta_o = \mu^{-1/3}z^{1/3}w_o$.

The path of steepest descents sweeps much farther into the lower half ζ plane than the path of steepest descents for the ion integral. The distance of the saddle point for given z from the origin is greater than in the case of the ion integral by a factor of $\mu^{-1/3}$. We perform calculations for cesium; in that case $\mu^{-1/3} \approx 62.4$. An unacceptably large number of residue contributions from the zeros of $[f^2K_+(\zeta)]$ must be included in a deformation to this contour from the path of steepest descents for the ion integral first adopted for both integrals. That portion of the path of steepest descents now being considered corresponding to $0 \leq t < +\infty$ may be used. The phase change of the integrand of the electron integral along it is $\sim 0(2\pi)$. Furthermore there are no zeros of $[f^2K_+(\zeta)]$ between it and the positive real axis.[7]

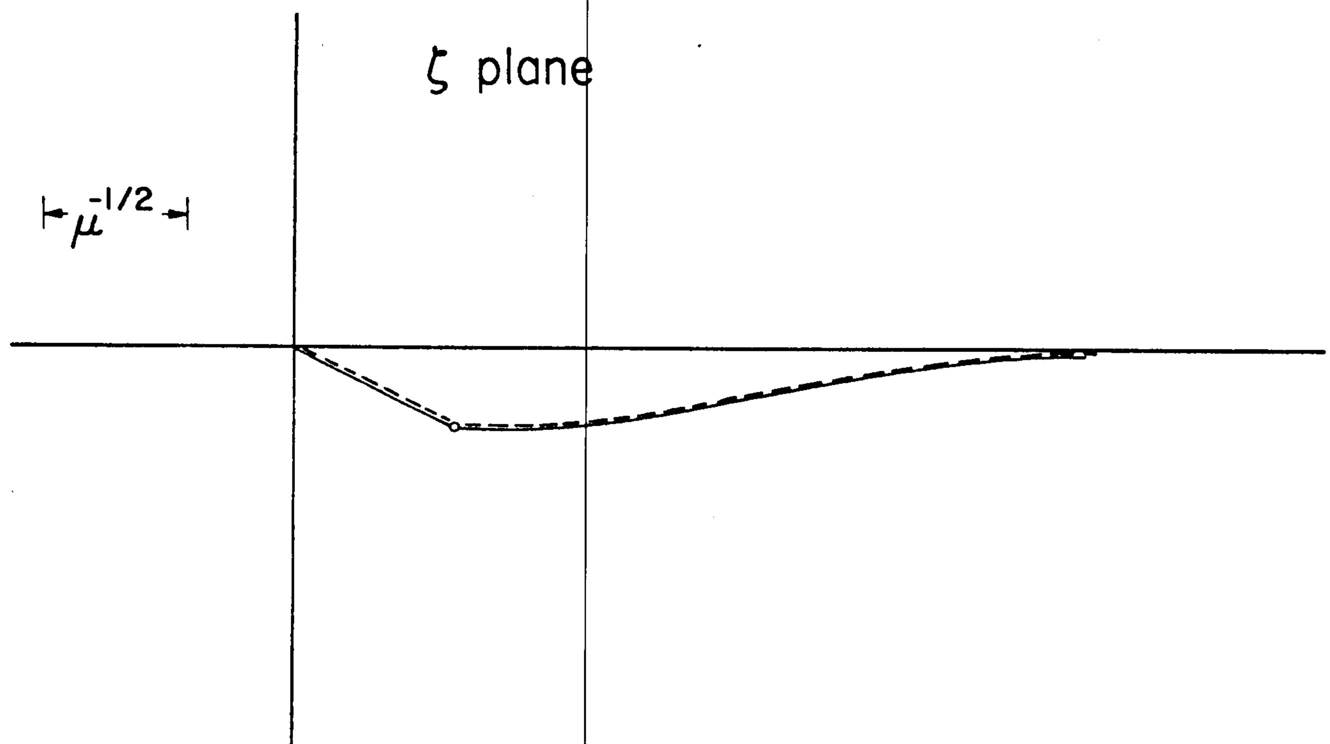
The completion of the contour for the electron integral by a straight line segment between the origin and $\zeta_o(=\mu^{-1/3}z^{1/3}w_o)$ requires the inclusion of the residue contribution of ζ_1 alone. See

Figure 7. Furthermore it well satisfies the criterion articulated in the preceding section that the phase change of the integrand is acceptably small. The argument of ζ on this contour, $-\pi/6$, gives substantial exponential damping as $|\zeta| \to 0$. We have $iz/\zeta = (z/|\zeta|)(-0.5 + .866\ i)$. As ζ moves along the straight line portion of the contour from a region where $z/|\zeta| \ll 1$ toward the origin, the phase of $e^{iz/\zeta}$ changes. There is, however, an associated change in the real part of the argument iz/ζ toward larger negative values, with the result that the phase of $e^{iz/\zeta}$ changes by no more than an amount of order 2π before the integrand becomes negligibly small.

As noted above, the fact that the ion and electron integrals differ only in the factors $e^{-\zeta^2}$ and $\mu^{1/2}e^{-\mu\zeta^2}$, respectively, makes it possible, independent of z, to begin the numerical integration of the electron integral at a value of ζ sufficiently large in absolute value ($|\zeta| \simeq 2.5$) that the phase change of the integral is further reduced. An additional consequence of this is that the functions $Z_{\pm}'(\zeta)$ may be approximated by their asymptotic expansions in the electron integral.

2.6 Numerical Results

The fundamental integrals for the dipole limit and for the finite grid separation case were numerically integrated on the TRW Systems On-Line Computer.[8] The case of a cesium plasma, for which $\mu = 4.12775 \times 10^{-6}$, was treated with $\tilde{T} = 1$ and $f^2 \ll 1$. All numerical inputs were taken rounded to six figures. A high accuracy integration routine was used. It is estimated that the results are



DEFORMED CONTOUR FOR ELECTRON INTEGRAL.

FIGURE 7.

accurate to three or possibly four places. Calculations of Φ_1/f^2 agree with the results of Gould.[4] See Figure 8. For the finite grid separation case, with the relations $\Phi = -\varepsilon_o\phi/\sigma_o x_o$ and $\Phi_2 = -\varepsilon_o\phi_2/\sigma_o x_o$, we have

$$\Phi = (2z_o)^{-1}\left[\Phi_2(z+z_o) - \Phi_2(|z+z_o|)\right] \tag{2.6.1}$$

in which $z_o = \omega_o x_o/2a_i$, and

$$\frac{\Phi_2}{f^2} = \frac{1}{2\pi}\int_0^\infty e^{iz/\zeta}\left[\frac{1}{f^2K_+(\zeta,f^2)} - \frac{1}{f^2K_-(\zeta,f^2)}\right]d\zeta . \tag{2.6.2}$$

This quantity was likewise calculated. See Table 1. For $z_o = 4$, $\arg(\Phi/f^2)$ and $\ln(|\Phi|/f^2)$ are shown in Figure 8 for points outside the grids. This is typical of the results for other values of z_o. The ion wave portion of the response, nearer the grid, is excited at a lower amplitude as z_o increases; there is an increase in the contribution of the branch-cut integral relative to the residue contribution. The electron wave at large distances is unaffected. The character of the response in the interference region, at intermediate distance, is affected. The selective modification of the response near the grid might have been expected, since the factor $[\sin(kx_o/2)/(kx_o/2)]$ reduces the contribution from large values of $|k|$ ($\gtrsim 2/x_o$) . The dominance of the residue contribution for a range of z is responsible for the roughly parallel shift of $\ln(|\Phi|/f^2)$ relative to $\ln(|\Phi_1|/f^2)$.

The numerical results obtained for the case of finite separation between the grids demonstrate that the nature of the response is determined primarily by the characteristics of the plasma, as embodied in the dielectric function, rather than by spatial characteristics of the excitation.

Table 1. Fundamental Integral for Finite Separation

z	$Re\{\Phi_2(z)/f^2\}$	$Im\{\Phi_2(z)/f^2\}$
1.0	$.378739 \times 10^{-1}$	$-.293679 \times 10^{-2}$
1.5	.276329	.914867
2.0	.177838	$.149934 \times 10^{-1}$
2.5	$.904137 \times 10^{-2}$	.167672
3.0	.187046	.157897
3.5	-.349902	.130614
4.0	-.689171	$.949861 \times 10^{-2}$
4.5	-.879190	.555833
5.0	-.924791	.180037
5.5	-.857435	-.143071
6.0	-.711236	-.392967
6.5	-.519392	-.561306
7.0	-.311467	-.649649
7.5	-.111600	-.666919
8.0	$.624318 \times 10^{-3}$	-.626820
8.5	$.199364 \times 10^{-2}$	-.545461
9.0	.293986	-.439280
9.5	.346247	-.323451
10.0	.360102	-.210754
10.5	.342246	-.110919
11.0	.300894	$-.303954 \times 10^{-3}$
11.5	.244682	.275234
12.0	.181785	.623190
12.5	.119264	.756945
13.0	$.626576 \times 10^{-3}$	.709402
13.5	.158075	.522945
14.0	-.191301	.243558
14.5	-.415192	$-.841016 \times 10^{-4}$
15.0	-.519780	$-.420444 \times 10^{-3}$
16.5	-.520570	-.733430
16.0	-.439015	-.999831

z	$\mathrm{Re}\{\Phi_2(z)/f^2\}$	$\mathrm{Im}\{\Phi_2(z)/f^2\}$
16.5	-.299339	$-.120544 \times 10^{-2}$
17.0	-.125786	-.137140
17.5	$.595241 \times 10^{-4}$	-.141798
18.0	$.238124 \times 10^{-3}$	-.143292
18.5	.396043	-.139968
19.0	.524086	-.133072
19.5	.617623	-.123898
20.0	.676016	-.113661
20.5	.701825	-.103410
21.0	.699891	$-.939721 \times 10^{-3}$
21.5	.676419	-.859238
22.0	.638143	-.795977
22.5	.591613	-.751052
23.0	.542667	-.723742
23.5	.496068	-.711962
24.0	.455323	-.712739
24.5	.422649	-.722667
25.0	.399061	-.738307
25.5	.384546	-.756492
26.0	.378294	-.774547
26.5	.378943	-.790421
27.0	.384822	-.802726
27.5	.394173	-.810714
28.0	.405321	-.814203
28.5	.416809	-.813463
29.0	.427479	-.809095
29.5	.436510	-.801901
30.0	.443414	-.792766
30.5	.448011	-.782561
31.0	.450372	-.772064
31.5	.450758	-.761915
32.0	.449557	-.752582
32.5	.447214	-.744362

z	$\mathrm{Re}\{\Phi_2(z)/f^2\}$	$\mathrm{Im}\{\Phi_2(z)/f^2\}$
33.0	.444186	-.737388
33.5	.440890	-.731653
34.0	.437678	-.727042
34.5	.434819	-.723367
35.0	.432491	-.720397
35.5	.430783	-.717893
36.0	.429710	-.715625
36.5	.429225	-.713398
37.0	.429235	-.711055
37.5	.429625	-.708490
38.0	.430265	-.705641
38.5	.431032	-.702491
39.0	.431816	-.699054
39.5	.432527	-.695374
40.0	.433102	-.691510
40.5	.433499	-.687527
41.0	.433703	-.683495
41.5	.433717	-.679474
42.0	.433561	-.675514
42.5	.433264	-.671656
43.0	.432861	-.667925
43.5	.432389	-.664333
44.0	.431883	-.660882
44.5	.431373	-.657565
45.0	.430884	-.654366
45.5	.430432	-.651269
46.0	.430027	-.648253
46.5	.429674	-.645300
47.0	.429371	-.642391
47.5	.429114	-.639512
48.0	.428895	-.636652
48.5	.428703	-.633804
49.0	.428531	-.630963

z	$Re\{\Phi_2(z)/f^2\}$	$Im\{\Phi_2(z)/f^2\}$
49.5	.428369	-.628127
50.0	.428209	-.625299
50.5	.428046	-.622480
51.0	.427876	-.619674
51.5	.427696	-.616886
52.0	.427505	-.614118
52.5	.427305	-.611375
53.0	.427079	-.608635
53.5	.426864	-.605964
54.0	.426649	-.603319
54.5	.426434	-.600698
55.0	.426219	-.598102
55.5	.426004	-.595529
56.0	.425789	-.592980
56.5	.425575	-.590454
57.0	.425360	-.587950
57.5	.425146	-.585468
58.0	.424931	-.583008
58.5	.424717	-.580569
59.0	.424503	-.578152
59.5	.424288	-.575755
60.0	.424074	-.573378
60.5	.423860	-.571022
61.0	.423646	-.568685
61.5	.423432	-.566367

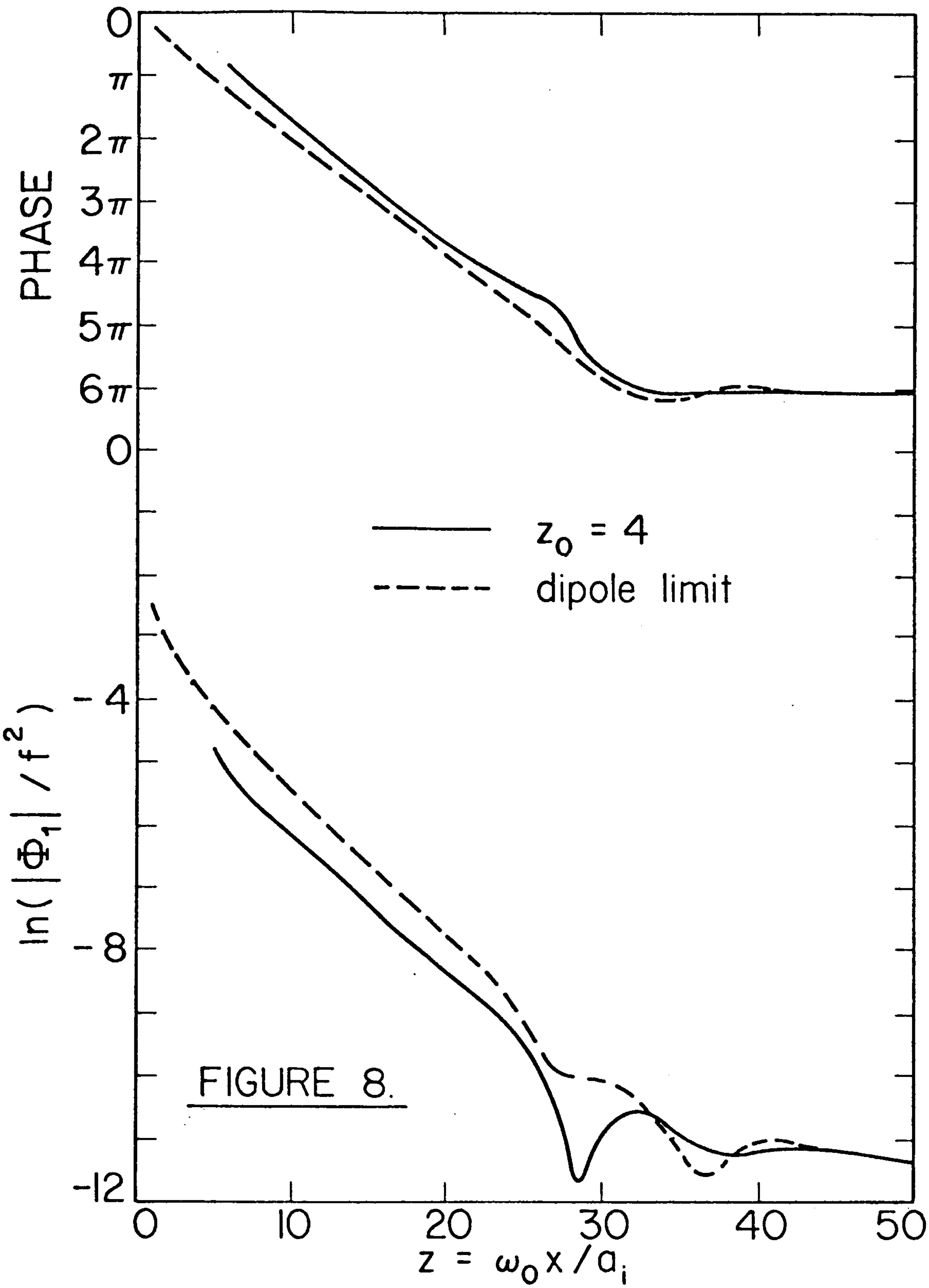


FIGURE 8.

NUMERICAL RESULTS IN LINEAR THEORY.

III. FORMULATION OF RESPONSE: NONLINEAR THEORY

We consider now the weakly nonlinear steady-state response to grid excitation. That is, we examine the case in which the amplitude of excitation is such that the perturbations of the species distribution functions, while still small compared with the unperturbed distribution functions, introduce nonlinearity into the Vlasov equation through the term involving the product of the electric field and the velocity gradient of the distribution function.

In Section 1 a perturbation series expansion of the potential and of the species distribution functions in the (nonlinear) Vlasov equation yields a hierarchy of equations associated with a smallness parameter proportional to the amplitude of grid excitation. In each order the equations are linear in the perturbation quantities of that order. In the first order the linearized Vlasov equation is obtained. In the equations of each order above the first there are driving terms composed of quadratic combinations of quantities of lower order. In the second order the steady-state response consists of zero frequency and double frequency components. In Section 2 the second order equations are Laplace-Fourier transformed in the manner of Landau.[2] The transform of the perturbation potential in second order is formally inverted; the inversion leads to a double integral over Fourier transform variables which is further complicated by the presence of branch-points in the plane of one Fourier transform variable whose position depends on the value of the other Fourier transform variable. The determination of the double frequency component of the lowest order nonlinear response thus appears unfeasible without some simplification. The desired simplification is described in Section 3. It is effected by approximating the steady-state potential in the linear theory, which appears quadratically in the driving term in second order, by the residue contribution which is dominant for values of the spatial variable neither too close to, nor too far from, the grid. The species distribution functions in the linear theory are found in this approximation. By calculating the species number density, $n_\alpha(x,t)$, and then "stripping off" the velocity integration, the deformation of contour of a subsequent velocity integration around the pole arising from the species distribution function is determined in advance. In Section 4 functions of two complex variables defined by velocity integrals in the lowest order nonlinear response are expressed in terms of plasma dispersion functions. Section 5 deals with the zero frequency component of the nonlinear response. It is shown that the zero frequency component of the nonlinear response is a polarization of the plasma unaccompanied by any zero frequency species current densities. These results are obtained independent of the perturbation expansion. The zero frequency component of the lowest order nonlinear response in the dominant pole approximation is obtained by residues.

3.1 Perturbation Expansion of Vlasov Equation

Examination of the (nonlinear) Vlasov equation suggests that steady-state response to time-harmonic grid excitation at frequency ω_o involves frequency components $\omega = 0$, ω_o, $2\omega_o$,.... Strong nonlinearity involves substantial contributions from a large number of harmonics and a coupled system which is probably difficult, if not impossible, to treat. Accordingly, a solution is sought by considering a perturbation expansion in powers of a smallness parameter proportional to the amplitude of the grid excitation.*

The conclusion reached earlier that the problem is one-dimensional is not changed if the Vlasov equation is nonlinear. Therefore, we begin with the one-dimensional Vlasov equation for excitation by a dipole grid

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)F_\alpha(x,v,t) - \frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial x}\Phi(x,t)\frac{\partial}{\partial v}F_\alpha(x,v,t) = 0 \qquad (3.1.1)$$

$$-\frac{\partial^2}{\partial x^2}\Phi(x,t) = -\frac{\sigma_0 x_0 d}{\epsilon_0 dx}\delta(x)\cos\omega_0 t + \sum_\alpha \frac{q_\alpha n_{0\alpha}}{\epsilon_0}\int_{-\infty}^{\infty} F_\alpha(x,v,t)\,dv. \qquad (3.1.2)$$

In the absence of excitation, we have $F_\alpha(x,v,t) = f_{0\alpha}(v)$.

Table 2 defines an appropriate set of dimensional variables, which are denoted by $\hat{F}_\alpha$, and so on. The dimensionless Vlasov equation is

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{v}\frac{\partial}{\partial \hat{x}}\right)\hat{F}_\alpha - \frac{\hat{\omega}_{p\alpha}^2}{\hat{n}_\alpha \hat{q}_\alpha}\frac{\partial}{\partial \hat{x}}\hat{\Phi}\frac{\partial}{\partial \hat{v}}\hat{F}_\alpha = 0 \qquad (3.1.3.)$$

$$-\frac{\partial^2}{\partial \hat{x}^2}\hat{\Phi} + \lambda\frac{d}{d\hat{x}}\delta(\hat{x})\cos\hat{t} - \sum_\alpha \hat{q}_\alpha \hat{n}_{0\alpha}\int_{-\infty}^{\infty}\hat{F}_\alpha\,d\hat{v} = 0 \qquad (3.1.4.)$$

---

* This is equivalent to the procedure used by Montgomery and Gorman[9] to study nonlinear Landau damping in time.

in which

$$\lambda = \frac{\sigma_o x_o}{e}\left(\frac{a_i}{\omega_o}\right) \tag{3.1.5}$$

is an appropriate smallness parameter for the expansion of potential and species distribution functions in perturbation series. The expansions are

$$\hat{F}_\alpha(\hat{x},\hat{v},t) = \hat{f}_{o\alpha}(\hat{v}) + \lambda\hat{f}_{1\alpha}(\hat{x},\hat{v},t) + \lambda^2\hat{f}_{2\alpha}(\hat{x},\hat{v},t) + \cdots \tag{3.1.6}$$

$$\hat{\Phi}(\hat{x},\hat{t}) = \qquad \lambda\hat{\phi}_1(\hat{x},\hat{t}) + \lambda^2\hat{\phi}_2(\hat{x},t) + \cdots \tag{3.1.7}$$

By substituting these expansions into Equations (3) and (4), and equation to zero separately the coefficient of λ^n for $n = 1,2,3,\ldots,$ there results a set of pairs of equations. For $n = 1$, the linearized Vlasov equation is recovered. For $n = 2,3,\ldots,$ the equations resulting from the Poisson equation are

$$\frac{\partial^2}{\partial\hat{x}^2}\hat{\phi}_n + \sum_\alpha \hat{q}_\alpha\hat{n}_{o\alpha}\int_{-\infty}^{\infty}\hat{f}_{n\alpha}d\hat{v} = 0. \tag{3.1.8}$$

Introducing the notation

$$\hat{\mathcal{L}}\hat{f}_{n\alpha} = \left(\frac{\partial}{\partial\hat{t}} + \hat{v}\frac{\partial}{\partial\hat{x}}\right)\hat{f}_{n\alpha} - \frac{\hat{\omega}_{p\alpha}^2}{\hat{n}_\alpha\hat{q}_\alpha}\frac{\partial}{\partial\hat{x}}\hat{\phi}_n\frac{\partial}{\partial\hat{v}}\hat{f}_{o\alpha} \tag{3.1.9}$$

the equations obtained from Equation (3) for $n = 2$ and 3, respectively, are

$$\hat{\mathcal{L}}\hat{f}_{2\alpha} = \frac{\hat{\omega}_{p\alpha}^2}{\hat{n}_\alpha\hat{q}_\alpha}\frac{\partial}{\partial\hat{x}}\hat{\phi}_1\frac{\partial}{\partial\hat{v}}\hat{f}_{1\alpha} \qquad (n=2) \tag{3.1.10}$$

Table 2. Dimensionless Variables

$$F_\alpha = a_i^{-1} \hat{F}_\alpha$$

$$t = \omega_o^{-1} \hat{t}$$

$$v = a_i \hat{v}$$

$$x = (a_i/\omega_o) \hat{x}$$

$$q_\alpha = e \hat{q}_\alpha \ (e > 0)$$

$$\Phi = (e\omega_o/\varepsilon_o a_i) \hat{\Phi}$$

$$n_{o\alpha} = (\omega_o/a_i)^3 \hat{n}_{o\alpha}$$

$$\omega_{p\alpha} = \omega_o \hat{\omega}_{p\alpha}$$

$$f_{o\alpha} = a_i^{-1} \hat{f}_{o\alpha}$$

$$\hat{d}\hat{f}_{3\alpha} = \frac{\hat{\omega}_{p\alpha}^2}{\hat{n}_\alpha q_\alpha}\left(\frac{\partial}{\partial \hat{x}}\hat{\phi}_2 \frac{\partial}{\partial \hat{v}}\hat{f}_{1\alpha} + \frac{\partial}{\partial \hat{x}}\hat{\phi}_1 \frac{\partial}{\partial \hat{v}}\hat{f}_{2\alpha}\right) \quad (n=3) \tag{3.1.11}$$

and so on for larger values of n . This set of pairs of equations provides a basis for determining the nonlinear steady-state response of the Vlasov equation to grid excitation for "sufficiently small" values of λ . We consider only the lowest order nonlinear equations, $n = 2$. The difficulty of treating these equations is considerable. The nonlinear equations of lowest order in terms of dimensional variables are

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\delta f_\alpha - \frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial x}\delta\phi\frac{\partial}{\partial v}f_{o\alpha} = -\frac{q_\alpha}{m_\alpha}E\frac{\partial}{\partial v}f_\alpha \tag{3.1.12}$$

$$-\frac{\partial^2}{\partial x^2}\delta\phi = \sum_\alpha \frac{q_\alpha n_{o\alpha}}{\epsilon_o}\int_{-\infty}^{\infty}\delta f_\alpha dv \tag{3.1.13}$$

in which $E = -\partial\phi/\partial x$. The potential and the species distribution functions, correct through second order, are

$$\Phi(x,t) = 0 + \phi(x,t) + \delta\phi(x,t) \tag{3.1.14}$$

$$F_\alpha(x,v,t) = f_{o\alpha}(v) + f_\alpha(x,v,t) + \delta f_\alpha(x,v,t) \; . \tag{3.1.15}$$

The potential and species distribution functions which satisfy the linearized Vlasov equation are $\phi(x,t)$ and $f_\alpha(x,v,t)$. The lowest order nonlinear contributions to the potential and the species distribution functions are $\delta\phi(x,t)$ and $\delta f_\alpha(x,v,t)$.

3.2 General Formulation of Lowest Order Nonlinear Response

Equations (3.1.12) and (3.1.13) are Laplace-Fourier transformed in the manner of Landau. In particular, we recall the requirement that ω be given a sufficiently large positive imaginary part to assure the existence of the Laplace transforms. With the operator notation

$$T = \int_0^\infty dt\, e^{i\omega t} \int_{-\infty}^{\infty} dx\, e^{-ikx} \tag{3.2.1}$$

there results

$$-i(\omega - kv)\delta f_\alpha(k,v,\omega) - \frac{q_\alpha}{m_\alpha} ik\delta\phi(k,\omega)\frac{\partial}{\partial v} f_{o\alpha}(v) = -\frac{q_\alpha}{m_\alpha} T\left(E\frac{\partial}{\partial v} f_\alpha\right) \tag{3.2.2}$$

$$k^2\delta\phi(k,\omega) = \sum_\alpha \frac{q_\alpha n_{o\alpha}}{\epsilon_o} \int_{-\infty}^{\infty} \delta f_\alpha(k,v,\omega)\,dv\,. \tag{3.2.3}$$

The Laplace-Fourier transform of the lowest order nonlinear perturbation to the potential is

$$\delta\phi(k,\omega) = \frac{i}{k^3\left[1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega/k)}\,dv\right]} \sum_\alpha \omega_{p\alpha}^2 \int_{-\infty}^{\infty} \frac{T\left(E\frac{\partial}{\partial v} f_{o\alpha}\right)}{(v-\omega/k)}\,dv. \tag{3.2.4}$$

The procedure for determining the steady-state behavior of $\delta\phi(x,t)$ without further approximations is now developed formally. The difficulty of evaluating the formal result will be abundantly apparent. The potential and the perturbations to the species distribution functions in the linearized theory are given in terms of their Laplace-Fourier representations

$$\phi(x',t') = \int_{L'} \frac{d\omega'}{2\pi} e^{-i\omega' t'} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} e^{ik'x'} \phi(k',\omega') \tag{3.2.5}$$

and

$$f_\alpha(x',v,t') = \int_{L''} \frac{d\omega''}{2\pi} e^{-i\omega'' t'} \int_{-\infty}^{\infty} \frac{dk''}{2\pi} e^{ik''x'} \frac{q_\alpha}{m_\alpha} \frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega''/k'')} \phi(k'',\omega''). \tag{3.2.6}$$

We are guided by the work of Landau to perform the velocity integration first. Hence, we define

$$G_\alpha\left(x',t';\frac{\omega}{k}\right) = \int_{-\infty}^{\infty} \frac{\left[-\frac{\partial}{\partial x'}\phi(x',t')\right]\frac{\partial}{\partial v} f_\alpha(x',v,t')}{(v-\omega/k)} dv \tag{3.2.7}$$

which is given more explicitly by

$$G_\alpha\left(x',t';\frac{\omega}{k}\right) = \int_{L'}\int_{L''} \frac{d\omega' d\omega''}{(2\pi)^2} e^{-i(\omega'+\omega'')t'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk'dk''}{(2\pi)^2} e^{i(k'+k'')x'}$$

$$\times \frac{q_\alpha}{m_\alpha}(-ik')\phi(k',\omega')\phi(k'',\omega'') \int_{-\infty}^{\infty} \frac{1}{(v-\omega/k)} \frac{\partial}{\partial v}\left[\frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega''/k'')}\right] dv. \tag{3.2.8}$$

The velocity integral must be analytically continued throughout the complex planes of its two arguments. We denote the resulting functions of the complex variables ω/k and ω''/k'' by $I_\alpha(\omega,k;\,\omega'',k'')$. We now perform the indicated Laplace inversions to obtain the steady-state behavior of $G_\alpha(x',t';\,\frac{\omega}{k})$. If

$$\phi(k,\omega) = \frac{i}{2(\omega-\omega_0)}\phi(k;\omega_0) + \frac{i}{2(\omega+\omega_0)}\phi(k;-\omega_0) \tag{3.2.9}$$

is the transform whose Laplace-Fourier inversion gives the steady-state behavior of $\phi(x,t)$ [see Equation (1.3.16)], the steady-state behavior of G_α is

$$\begin{aligned} G_\alpha(x',t';\tfrac{\omega}{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk'dk''}{(2\pi)^2} e^{i(k'+k'')x'} \frac{q_\alpha}{m_\alpha}(-ik') \\ \times\Big\{\frac{1}{4}e^{-2i\omega_0 t'}\phi(k';\omega_0)\phi(k'';\omega_0)I_\alpha(\omega,k;\omega_0,k'') \\ +\frac{1}{4}e^{2i\omega_0 t'}\phi(k';-\omega_0)\phi(k'';-\omega_0)I_\alpha(\omega,k;-\omega_0,k'') \\ +\frac{1}{4}\Big[\phi(k';\omega_0)\phi(k'';-\omega_0)I_\alpha(\omega,k;-\omega_0,k'') \\ +\phi(k';-\omega_0)\phi(k'';\omega_0)I_\alpha(\omega,k;\omega_0,k'')\Big]\Big\} . \end{aligned} \tag{3.2.10}$$

The Laplace-Fourier transform of this quantity is

$$\int_C \frac{T\left(-\frac{\partial\phi}{\partial x}\frac{\partial f_{0\alpha}}{\partial v}\right)}{(v-\omega/k)}dv = \int_{-\infty}^{\infty}dx' e^{-ikx'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk'dk''}{(2\pi)^2}e^{i(k'+k'')x'}\frac{q_\alpha}{m_\alpha}(-ik')$$

$$\times\left\{\frac{1}{4}\frac{i}{(\omega-2\omega_0)}\phi(k';\omega_0)\,\phi(k'';\omega_0)\,I_\alpha(\omega,k;\omega_0,k'')\right.$$

$$+\frac{1}{4}\frac{i}{(\omega+2\omega_0)}\phi(k';-\omega_0)\,\phi(k'';-\omega_0)\,I_\alpha(\omega,k;-\omega_0,k'')$$

$$+\frac{1}{4}\frac{i}{\omega}\Big[\phi(k';\omega_0)\,\phi(k'';-\omega_0)\,I_\alpha(\omega,k;-\omega_0,k'')$$

$$\left.+\,\phi(k';-\omega_0)\,\phi(k'';\omega_0)\,I_\alpha(\omega,k;\omega_0,k'')\Big]\right\}. \qquad (3.2.11)$$

The contour of integration C is deformed as necessary for the analytic continuation of functions defined by velocity integrals. Using the relation

$$\int_{-\infty}^{\infty} e^{-i(k-k'-k'')x'}\,dx' = 2\pi\,\delta(k-k'-k'') \qquad (3.2.12)$$

and performing the integration with respect to k" , we have

$$\int_C \frac{T\left(-\frac{\partial\phi}{\partial x}\frac{\partial f_{o\alpha}}{\partial v}\right)}{(v-\omega/k)}dv = \int_{-\infty}^{\infty}\frac{dk'}{2\pi}\frac{q_\alpha}{m_\alpha}(-ik')\left\{\frac{1}{4}\frac{i}{(\omega-2\omega_0)}\phi(k';\omega_0)\,\phi(k-k';\omega_0)\,I_\alpha(\omega,k;\omega_0,k-k')\right.$$

$$+\frac{1}{4}\frac{i}{(\omega+2\omega_0)}\phi(k';-\omega_0)\,\phi(k-k';\omega_0)\,I_\alpha(\omega,k;-\omega_0,k-k')$$

$$+\frac{1}{4}\frac{i}{\omega}\Big[\phi(k';\omega_0)\,\phi(k-k';-\omega_0)\,I_\alpha(\omega,k;-\omega_0,k-k')$$

$$\left.+\,\phi(k';-\omega_0)\,\phi(k-k';\omega_0)\,I_\alpha(\omega,k;\omega_0,k-k')\Big]\right\}. \qquad (3.2.13)$$

It is not necessary to proceed further to appreciate the difficulty of determining the steady-state behavior of $\delta\phi(x,t)$ by the Laplace-Fourier inversion of the analytic continuation of Equation (4) with the substitution of Equation (13). First, there is a double integration over Fourier transform variables. Second, the integrand contains functions of the arguments k' and $(k - k')$ which are different functions depending upon the sign of each argument on the primitive inversion contours of the transform variables. The location of one of these transition points (pairs of branch-points) in the plane of one transform variable depends upon the value of the other transform variable. We now describe an approximate method which permits us to avoid this impasse.

3.3 Dominant Pole Approximation

The steady-state behavior of $\delta\phi(x,t)$ may be determined by a program whose central notion is the approximation of the steady-state potential in the linearized theory, $\phi(x,t)$, by the residue contribution of the pole at $\zeta = \zeta_1$. The steady-state behavior of $f_\alpha(x,v,t)$ corresponding to this potential is then determined. We recall that a velocity integral is to be performed in obtaining $\delta\phi(k,\omega)$ and that, in accordance with the approach of Landau,[2] singularities in functions defined by velocity integrals are to be avoided by deformation of the contour of integration. This imposes the requirement that each velocity denominator in $f_\alpha(x,v,t)$ be scrutinized concerning the orientation of a subsequent velocity integration contour relative to the singularity in the complex velocity plane which it produces. This is achieved conveniently by determining the perturbed species number densities, which are velocity integrals of the species distribution functions, and by "stripping off" the velocity integration. The Laplace-Fourier transform of $\left(-\frac{\partial\phi}{\partial x}\frac{\partial f_\alpha}{\partial v}\right)$ is obtained in closed form. Then the other operations involved in determining the steady-state behavior of $\delta\phi(x,t)$ are performed. A single Fourier inversion integral is obtained which is of the same general character as that obtained in the linearized theory.

The dominant pole approximation for the electric field in the linearized theory is

$$-\frac{\partial}{\partial x}\phi(x,t) = f^2 A e^{i(k_1|x|-\omega_0 t)} + \text{c.c.} \qquad (-\infty < x < \infty) \tag{3.3.1}$$

in which

$$A = \frac{\sigma_0 x_0}{2\epsilon_0}\,\frac{i\omega_0}{a_i}\,\frac{1}{s_1^2\left[f^2 K_+'(s_1)\right]} \quad . \tag{3.3.2}$$

This is obtained from the steady-state potential $\phi_1(x,t) = (e^{-i\omega_0 t}/2) \times (-\sigma_0 x_0/\varepsilon_0)\,\Phi_1 + \text{c.c.}$ by approximating Φ_1 of Equation (2.2.2) by the residue contribution obtained when the integration contour is deformed to the path of steepest descents for the ion integral.

As was stated in the summary description of the dominant pole approximation above, the steady-state behavior of the species perturbation number densities will now be obtained. The integration with respect to velocity will then be "stripped off", note being taken of the contour of integration in anticipation of the subsequent velocity integration to be performed. The explicit form of Equation (3.2.9) is

$$\phi(k,\omega) = \left(-i\frac{\sigma_0 x_0}{2\epsilon_0}\right)\left[\frac{i}{(\omega-\omega_0)}\frac{1}{kK(k,\omega_0)} + \frac{i}{(\omega+\omega_0)}\frac{1}{kK(k,-\omega_0)}\right]. \tag{3.3.3}$$

Introducing this expression into

$$f_\alpha(k,v,\omega) = \frac{q_\alpha}{m_\alpha}\frac{\partial f_{0\alpha}(v)/\partial v}{(v-\omega/k)}\phi(k,\omega), \tag{3.3.4}$$

which is obtained from Equation (1.3.11), integrating with respect to v , analytically continuing the velocity integral, and performing the Laplace inversion, there results

$$n_\alpha(x,t) =$$

$$e^{-i\omega_0 t}\left(-i\frac{\sigma_0 x_0}{2\epsilon_0}\frac{q_\alpha}{m_\alpha}\right)\left[\int_{-\infty}^{0}\frac{dk}{2\pi}\frac{e^{ikx}}{kK_-(k,\omega_0)}\frac{1}{a_\alpha^2}Z_-'\left(\frac{\omega_0}{ka_\alpha}\right) + \int_0^\infty\frac{dk}{2\pi}\frac{e^{ikx}}{kK_+(k,\omega_0)}\frac{1}{a_\alpha^2}Z_+'\left(\frac{\omega_0}{ka_\alpha}\right)\right] +$$

$$e^{i\omega_0 t}\left(-i\frac{\sigma_0 x_0}{2\epsilon_0}\frac{q_\alpha}{m_\alpha}\right)\left[\int_{-\infty}^{0}\frac{dk}{2\pi}\frac{e^{ikx}}{kK_-(k,-\omega_0)}\frac{1}{a_\alpha^2}Z_-'\left(\frac{-\omega_0}{ka_\alpha}\right) + \int_0^\infty\frac{dk}{2\pi}\frac{e^{ikx}}{kK_+(k,-\omega_0)}\frac{1}{a_\alpha^2}Z_+'\left(\frac{-\omega_0}{ka_\alpha}\right)\right]. \tag{3.3.5}$$

The second term is the complex conjugate of the first, as can be seen with the aid of the identity

$$\left[Z_\pm(\zeta)\right]^* = Z_\mp(\zeta^*) \tag{3.3.6}$$

and the differential equation[5]

$$Z_\pm'(\zeta) = -2 - 2\zeta Z_\pm(\zeta). \tag{3.3.7}$$

For $x > 0$ and $f^2 \ll 1$, therefore, the dominant pole approximation is

$$n_\alpha(x,t) = f^2 e^{-i\omega_0 t}\left(\frac{\sigma_0 x_0}{2\epsilon_0}\frac{q_\alpha}{m_\alpha}\right)\frac{e^{ik_1 x}}{\zeta_1[f^2 K_+'(\zeta_1)]}\frac{1}{a_\alpha^2}Z_+'(\mu_\alpha^{1/2}\zeta_1) + \text{c.c.} \tag{3.3.8}$$

where $\mu_\alpha^{1/2} = 1$ for $\alpha = i$ and $\mu^{1/2}$ for $\alpha = e$. Treatment of the case $x < 0$ requires use of the identity

$$Z_\pm'(-\zeta) = Z_\mp'(\zeta) \tag{3.3.9}$$

in relating K_+ and K_-; this identity is most easily established from Equations (1.4.5) - (1.4.7). Putting $x = -X$ $(X > 0)$, with the transformation $k = -\kappa$, there results

$$n_\alpha(-X,t) = f^2 e^{-i\omega_0 t}\left(i\frac{\sigma_0 x_0}{2\epsilon_0}\frac{q_\alpha}{m_\alpha}\right)$$

$$\times\left[\int_0^{\infty}\frac{d\kappa}{2\pi}\frac{e^{i\kappa X}}{\kappa K_+(\kappa,\omega_0)}Z_-'\left(\frac{\omega_0}{-\kappa a_\alpha}\right) + \int_{-\infty}^{0}\frac{d\kappa}{2\pi}\frac{e^{i\kappa X}}{\kappa K_-(\kappa,\omega_0)}Z_+'\left(\frac{\omega_0}{-\kappa a_\alpha}\right)\right] + \text{c.c.} \tag{3.3.10}$$

In the single pole approximation, therefore, for $x < 0$ and $f^2 \ll 1$,

$$n_\alpha(x,t) = -f^2 e^{-i\omega_0 t}\left(\frac{\sigma_0 x_0}{2\epsilon_0}\frac{q_\alpha}{m_\alpha}\right)\frac{e^{-ik_1 x}}{\zeta_1[f^2 K_+'(\zeta_1)]}\frac{1}{a_\alpha^2}Z_-'(-\mu_\alpha^{1/2}\zeta_1) + \text{c.c.} \tag{3.3.11}$$

Combining Equations (8) and (11), and making use of Equations (6) and (7), we have the dominant pole approximation for the perturbation to the species number density in the linearized theory,

$$n_\alpha(x,t) =$$

$$f^2\frac{q_\alpha}{m_\alpha}Be^{i(k_1|x|-\omega_0 t)}\left[\theta(x)\frac{1}{a_\alpha^2}Z_+'(\mu_\alpha^{1/2}\zeta_1)-\theta(-x)\frac{1}{a_\alpha^2}Z_-'(-\mu_\alpha^{1/2}\zeta_1)\right] +$$

$$f^2\frac{q_\alpha}{m_\alpha}B^*e^{-i(k_1^*|x|-\omega_0 t)}\left[\theta(x)\frac{1}{a_\alpha^2}Z_-'(\mu_\alpha^{1/2}\zeta_1^*)-\theta(-x)\frac{1}{a_\alpha^2}Z_+'(-\mu_\alpha^{1/2}\zeta_1^*)\right] \tag{3.3.12}$$

in which $\theta(x)$ is the unit step function and

$$B = -\frac{\sigma_0 x_0}{2\epsilon_0}\frac{1}{\zeta_1[f^2K_+'(\zeta_1)]} \; . \tag{3.3.13}$$

The dominant pole approximation for $f_\alpha(x,v,t)$ is therefore

$$f_\alpha(x,v,t) =$$

$$f^2\frac{q_\alpha}{m_\alpha}Be^{i(k_1|x|-\omega_0 t)}\left[\theta(x)\frac{\partial f_{0\alpha}/\partial v}{(v-\zeta_1 a_\alpha)_+}-\theta(-x)\frac{\partial f_{0\alpha}/\partial v}{(v+\zeta_1 a_\alpha)_-}\right] +$$

$$f^2\frac{q_\alpha}{m_\alpha}B^*e^{-i(k_1^*|x|-\omega_0 t)}\left[\theta(x)\frac{\partial f_{0\alpha}/\partial v}{(v-\zeta_1^* a_\alpha)_-}-\theta(-x)\frac{\partial f_{0\alpha}/\partial v}{(v+\zeta_1^* a_\alpha)_+}\right] . \tag{3.3.14}$$

The $+$ and $-$ subscripts on the velocity denominators give the proper orientation of a subsequent velocity integration relative to the associated singularity. In all cases the contour indicated is one which deviates from the real axis to surround the singularity.

The equations for $\delta\phi(x,t)$ and $\delta f_\alpha(x,v,t)$ form a set of linear partial differential equations driven by the real quantities $(-\,\partial\phi/\partial x)(\partial f_\alpha/\partial v)$ which are of the form

$$-\frac{\partial}{\partial x}\phi\frac{\partial}{\partial v}f_\alpha = e^{-2i\omega_0 t}S_\alpha^{(+)}(x,v) + e^{2i\omega_0 t}S_\alpha^{(-)}(x,v) + S_\alpha^{(0)}(x,v) \quad . \tag{3.3.15}$$

Here $S_\alpha^{(-)}(x,v) = [S_\alpha^{(+)}(x,v)]^*$ and $S_\alpha^{(0)}(x,v)$ is real. The nonlinear potential $\delta\phi(x,t)$ is related to $(-\partial\phi/\partial x)(\partial f_\alpha/\partial v)$ by the Laplace-Fourier inversion of Equation (3.2.4). The Laplace transform-inversion procedure involved in obtaining the steady state result is trivial. The operator T induces

$$e^{\mp 2i\omega_0 t} \longrightarrow \frac{i}{(\omega \mp 2\omega_0)} \quad . \tag{3.3.16}$$

Considering the steady state, the Laplace inversion reverses the indicated transformation and produces the substitution $\omega \to \pm 2\omega_0$. Hence, the final result is of the form

$$\delta\phi(x,t) = e^{-2i\omega_0 t}S^{(+)}(x) + e^{2i\omega_0 t}S^{(-)}(x) + S^{(0)}(x) \tag{3.3.17}$$

in which the first term is the result of introducing in Equation (3.2.4) the partial excitation given by

$$-\frac{\partial}{\partial x}\phi(x,t)\frac{\partial}{\partial v}f_\alpha(x,v,t) = e^{-2i\omega_0 t}S_\alpha^{(+)}(x,v) \quad . \tag{3.3.18}$$

In order to obtain the double frequency response, therefore, it is sufficient to consider this partial excitation.

3.4 Relation of Velocity Integrals to Plasma Dispersion Functions

The functions defined by the analytic continuation of $I_\alpha(\omega,k;\ \omega'',k'')$ may be expressed in terms of the functions $Z_\pm(\zeta)$. For clarity, we repeat the definition

$$I_\alpha(\omega,k;\omega'',k'')=\int_{-\infty}^{\infty}\frac{1}{(v-\omega/k)}\frac{\partial}{\partial v}\left[\frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega''/k'')}\right]dv\ . \tag{3.4.1}$$

Integrating by parts, we have

$$I_\alpha=-\int_{-\infty}^{\infty}\frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega''/k'')}\frac{\partial}{\partial v}\left[\frac{1}{(v-\omega/k)}\right]dv \tag{3.4.2}$$

which in turn may be expressed as

$$I_\alpha=\frac{\partial}{\partial(\omega/k)}\int_{-\infty}^{\infty}\frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega/k)(v-\omega''/k'')}dv\ . \tag{3.4.3}$$

Effecting a separation into partial fractions, we have

$$I_\alpha=\frac{\partial}{\partial(\omega/k)}\left\{\frac{1}{\left(\frac{\omega''}{k''}-\frac{\omega}{k}\right)}\left[\int_{-\infty}^{\infty}\frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega''/k'')}dv-\int_{-\infty}^{\infty}\frac{\partial f_{o\alpha}(v)/\partial v}{(v-\omega/k)}dv\right]\right\}\ . \tag{3.4.4}$$

Defining $\zeta_\alpha = \omega/ka_\alpha$ and $\zeta_\alpha'' = \omega''/k''a_\alpha$ and analytically continuing the integrals, we obtain

$$I_\alpha=\frac{1}{a_\alpha^4}\frac{\partial}{\partial\zeta_\alpha}\left\{\frac{1}{(\zeta_\alpha''-\zeta_\alpha)}\left[Z'(\zeta_\alpha'')-Z'(\zeta_\alpha)\right]\right\}\ . \tag{3.4.5}$$

The Z' functions are Z'_+ or Z'_-, depending upon the sign of k

or k" on its primitive Fourier inversion contour. Having associated with each velocity denominator in the dominant pole approximation a plus or minus subscript, we have determined which of the two functions $Z'_{\pm}(\zeta_\alpha'')$ is to be chosen in each case. (The arguments ζ_α'' are constants in the analysis based upon the dominant pole approximation.)

For convenience, we define

$$V_{s s''}(\zeta_\alpha, \zeta_\alpha'') = a_\alpha^4 I_\alpha(\omega, k; \omega'', k'') \; . \qquad (3.4.6)$$

The subscript s(s") is plus or minus according to whether the Z function of argument $\zeta_\alpha(\zeta_\alpha'')$ is a plus or a minus function. Sometimes the first subscript will be omitted when V appears in an integral with respect to k , just as K is sometimes used to denote $K_{\pm}$, whichever is appropriate. An empty pair of parentheses is used to indicate omitted subscripts. The functions V may also be expressed as

$$V_{s s'}(\zeta_\alpha, \zeta_\alpha'') = \frac{[Z'_{s''}(\zeta_\alpha'') - Z'_s(\zeta_\alpha)]}{(\zeta_\alpha - \zeta_\alpha'')^2} + \frac{Z''_s(\zeta_\alpha)}{(\zeta_\alpha - \zeta_\alpha'')} \; . \qquad (3.4.7)$$

3.5 Zero Frequency Response

The nonlinear response at zero frequency is now shown to be a polarization of the plasma with no associated species current densities. This result, which is established independently of the perturbation expansion, is important because steady state currents would remove the plasma. The zero frequency component of the lowest order nonlinear contribution to the potential is determined by residues.

The absence of steady species current densities associated with large amplitude excitation may be shown in two ways. First we note that the zero frequency potential must be an even function of x because the charge density produced by the grid pair satisfies the relation $\rho_e(x,t + \pi/\omega_o) = \rho_e(-x,t)$ and hence there is no preferred direction relative to the grid when the excitation is time-averaged over one period. The only possible source of a zero frequency potential in the plasma is the inequality of the charge-to-mass ratio for the two species, which would lead to a potential which is an even function of x. Hence, there cannot be any zero frequency species current densities at $x = 0$. The Vlasov equation implies species continuity equations, which are obtained as velocity integrals of Equation (1.2.1), namely

$$\frac{\partial}{\partial t} n_\alpha + \frac{\partial}{\partial \underline{x}} \cdot \underline{\Gamma}_\alpha = 0 \ , \tag{3.5.1}$$

in which $\underline{\Gamma}_\alpha$ is the particle current density of species α. For the zero frequency component this equation states that the species particle current densities are divergenceless; since they are zero at $x = 0$ they must be zero everywhere. This result is independent of the perturbation expansion.

The second method of demonstrating the absence of species particle current densities is by explicit calculation, and is therefore valid only to lowest nonlinear order in the perturbation expansion. From Equation (3.2.2), solving for $\delta f_\alpha(k,v,\omega)$, multiplying by v, substituting for $f_\alpha(x,v,t)$ from Equation (3.3.4), and integrating with respect to v , we have

$$\delta\Gamma_\alpha(k,\omega) = \frac{q_\alpha}{m_\alpha}\delta\phi(k,\omega)\int_{-\infty}^{\infty}\frac{v\,\partial f_{o\alpha}(v)/\partial v}{(v-\omega/k)}dv$$

$$-\frac{q_\alpha}{m_\alpha}\frac{1}{ik}\int_0^\infty dt' e^{i\omega t'}\int_{-\infty}^{\infty}dx' e^{-ikx'}\int_{L'}\int_{L''}\frac{d\omega' d\omega''}{(2\pi)^2}e^{-i(\omega'+\omega'')t'}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk'dk''}{(2\pi)^2}e^{i(k'+k'')x'}(-ik')\phi(k',\omega')\phi(k'',\omega'')\int_{-\infty}^{\infty}\frac{v}{(v-\omega/k)}\frac{\partial}{\partial v}\left[\frac{\partial f_{o\alpha}/\partial v}{(v-\omega''/k'')}\right]dv . \qquad (3.5.2)$$

When $\omega = 0$ the two velocity integrals vanish so that $\delta\Gamma_\alpha(k,0) = 0$. Hence there are no zero frequency species particle current densities.

We now determine the spatial behavior of the lowest order zero frequency potential in the dominant pole approximation. Substituting from Equations (3.3.1) and (3.3.14), the quantity $S_\alpha^{(o)}(x,v)$ of Equation (3.3.15) is

$$S_\alpha^{(0)}(x,v) =$$

$$f^4\frac{q_\alpha}{m_\alpha}\theta(x)e^{-2k_I x}\left\{A^*B\frac{\partial}{\partial v}\left[\frac{\partial f_{o\alpha}/\partial v}{(v-\varsigma_1 a_i)_+}\right]+AB^*\frac{\partial}{\partial v}\left[\frac{\partial f_{o\alpha}/\partial v}{(v-\varsigma_1^* a_i)_-}\right]\right\}$$

$$-f^4\frac{q_\alpha}{m_\alpha}\theta(-x)e^{2k_I x}\left\{A^*B\frac{\partial}{\partial v}\left[\frac{\partial f_{o\alpha}/\partial v}{(v+\varsigma_1 a_i)_-}\right]+AB^*\frac{\partial}{\partial v}\left[\frac{\partial f_{o\alpha}/\partial v}{(v+\varsigma_1^* a_i)_+}\right]\right\} \qquad (3.5.3)$$

in which $k_I = \mathrm{Im}\{k_1\}$. The Laplace-Fourier transform of this quantity, $S_\alpha^{(o)}(k,v,\omega) = T[S_\alpha^{(o)}(x,v)]$, is

$$S_\alpha^{(0)}(k,v,\omega)=\frac{i}{\omega}f^4\frac{q_\alpha}{m_\alpha}\frac{1}{i}\Big\{\Big\}_>\frac{1}{(k-2ik_I)}+\frac{i}{\omega}f^4\frac{q_\alpha}{m_\alpha}\frac{1}{i}\Big\{\Big\}_<\frac{1}{(k+2ik_I)} \tag{3.5.4}$$

in which > (<) refers to the first (second) curly bracket in Equation (3). Introducing this expression into Equation (3.2.4), analytically continuing the velocity integrals in the dielectric function, and performing the Laplace-Fourier inversion, the zero frequency component of the potential is [see Equation (3.3.17)] given by

$$\begin{aligned}\frac{S^{(0)}(x)}{f^4}=&\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{ie^{ikx}}{k^3K(k,0)(k-2ik_I)}\sum_\alpha U_\alpha\\ &\times\left[\left(\frac{A^*B}{i}\right)V_{()+}(0,\mu_\alpha^{1/2}\zeta_1)+\left(\frac{AB^*}{i}\right)V_{()-}(0,\mu_\alpha^{1/2}\zeta_1^*)\right]\\ &+\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{ie^{ikx}}{k^3K(k,0)(k+2ik_I)}\sum_\alpha U_\alpha\\ &\times\left[\left(\frac{A^*B}{i}\right)V_{()-}(0,-\mu_\alpha^{1/2}\zeta_1)+\left(\frac{AB^*}{i}\right)V_{()+}(0,-\mu_\alpha^{1/2}\zeta_1^*)\right]\end{aligned} \tag{3.5.5}$$

in which $U_\alpha = \omega_{p\alpha}^2 q_\alpha/m_\alpha a_\alpha^4$. In the static limit, because of the identity $Z'_\pm(0) = -2$, the plus and minus dielectric functions are equal, i.e., $K_\pm(k,0) = 1 + (k_D^2/k^2)$, where k_D is defined after Equation (2.1.2). The coefficients of the V's are

$$\left(\frac{A^*B}{i}\right)=\left(\frac{\sigma_0 x_0}{2\epsilon_0}\right)^2\frac{\omega_0}{a_i}\frac{1}{|\zeta_1 f^2 K_+'(\zeta_1)|^2}\frac{1}{\zeta_1^*} \tag{3.5.6}$$

$$\left(\frac{AB^*}{i}\right) = -\left(\frac{\sigma_o x_o}{2\epsilon_o}\right)^2 \frac{\omega_o}{a_i} \frac{1}{|\zeta_1 f^2 K_+'(\zeta_1)|^2} \frac{1}{\zeta_1} \quad . \tag{3.5.7}$$

Making use of the identities $Z'_{\pm}(0) = -2$ and $Z''_{\pm}(0) = \mp 2\sqrt{\pi}\, i$ [from Equations (2.2.4) and (3.3.7)] in Equation (3.4.7), we have

$$V_{++}(0,\mu_\alpha^{1/2}\zeta_1) = \left[\frac{Z_+'(\mu_\alpha^{1/2}\zeta_1)+2}{\mu_\alpha \zeta_1^2} \pm \frac{2\sqrt{\pi}\, i}{\mu_\alpha^{1/2}\zeta_1}\right] \tag{3.5.8}$$

$$V_{\pm-}(0,\mu_\alpha^{1/2}\zeta_1^*) = \left[\frac{Z_-'(\mu_\alpha^{1/2}\zeta_1^*)+2}{\mu_\alpha \zeta_1^{*2}} \pm \frac{2\sqrt{\pi}\, i}{\mu_\alpha^{1/2}\zeta_1^*}\right] \tag{3.5.9}$$

$$V_{\pm-}(0,-\mu_\alpha^{1/2}\zeta_1) = \left[\frac{Z_-'(-\mu_\alpha^{1/2}\zeta_1)+2}{\mu_\alpha \zeta_1^2} \mp \frac{2\sqrt{\pi}\, i}{\mu_\alpha^{1/2}\zeta_1}\right] \tag{3.5.10}$$

$$V_{\pm+}(0,-\mu_\alpha^{1/2}\zeta_1^*) = \left[\frac{Z_+'(-\mu_\alpha^{1/2}\zeta_1^*)+2}{\mu_\alpha \zeta_1^{*2}} \mp \frac{2\sqrt{\pi}\, i}{\mu_\alpha^{1/2}\zeta_1^*}\right] \tag{3.5.11}$$

Using Equations (3.3.6), (3.3.7), (6), (7), (8), and (9), we have

$$[\;]_> = 2i\left(\frac{\sigma_o x_o}{2\epsilon_o}\right)^2 \frac{\omega_o}{a_i} \frac{1}{|\zeta_1^2 f^2 K_+'(\zeta_1)|^2} \mathrm{Im}\left\{\frac{Z_+'(\mu_\alpha^{1/2}\zeta_1)+2}{\mu_\alpha \zeta_1}\right\} \tag{3.5.12}$$

where $[\;]_>$ denotes the first square bracket in Equation (3.5.5). Using Equations (3.3.6), (3.3.7), (3.3.9), (6), (7), (10), and (11), we determine that $[\;]_<$ equals $[\;]_>$. The plus and minus functions are equal and we can evaluate Equation (5) by residues. Suppressing an obvious group of constants, the integral to be evaluated is

(neglecting Debye shielding poles)

$$I = \frac{i}{k_D^2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{k}\left[\frac{1}{(k-2ik_I)}+\frac{1}{(k+2ik_I)}\right] . \tag{3.5.13}$$

Evaluating by residues, we have

$$I = \frac{i}{2k_D^2}\frac{1}{k_I}e^{-2k_I|x|} . \tag{3.5.14}$$

Hence, we obtain

$$\frac{S^{(0)}(x)}{f^4} = -\left(\frac{\sigma_0 x_0}{2\epsilon_0}\right)^2\frac{\omega_0}{a_i}\frac{1}{|\zeta_1^2 f^2 K_+'(\zeta_1)|^2 k_D^2 k_I}$$
$$\times\left\{U_i \,\mathrm{Im}\left[\frac{Z_+'(\zeta_1)+2}{\zeta_1}\right]+U_e \,\mathrm{Im}\left[\frac{Z_+'(\mu^{1/2}\zeta_1)+2}{\mu\,\zeta_1}\right]\right\}e^{-2k_I|x|} . \tag{3.5.15}$$

For a plasma consisting of electrons and singly charged ions at the same temperature $U_i = U = -U_e$, where

$$U = \frac{n e^3}{\epsilon_0(2KT)^2} . \tag{3.5.16}$$

With the power series expansion[5]

$$Z_+'(\zeta) = -2 - 2\sqrt{\pi}\, i\, \zeta e^{-\zeta^2} + \cdots \tag{3.5.17}$$

Equation (15) becomes

$$\frac{S^{(0)}(x)}{f^4} = -\left(\frac{\sigma_0 x_0}{2\epsilon_0}\right)^2 \frac{\omega_0}{a_i} \frac{U}{|\zeta_1^2 f^2 K_+'(\zeta_1)|^2 k_D^2 k_I}$$

$$\times \left\{ Im\left[\frac{Z_+'(\zeta_1)+2}{\zeta_1}\right] + \frac{2\sqrt{\pi}}{\mu^{1/2}} \right\} e^{-2k_I|x|} . \tag{3.5.18}$$

Since $\mu^{1/2} << 1$, using the definition of k_D , we finally obtain

$$\frac{S^{(0)}(x)}{f^4} = -\frac{\sqrt{\pi}}{2\mu^{1/2}}\left(\frac{e^2\omega_0}{m_i a_i^3}\right)\left(\frac{\sigma_0 x_0}{2\epsilon_0}\right)^2 \frac{1}{k_I|\zeta_1^2 f^2 K_+'(\zeta_1)|^2} e^{-2k_I|x|} \tag{3.5.19}$$

as the zero frequency component of the lowest order nonlinear potential in the plasma.

IV. DOUBLE FREQUENCY RESPONSE

We now consider the response obtained from the Laplace-Fourier inversion of Equation (3.2.4) with the partial excitation given by Equation (3.3.18), which was shown to be adequate to determine the double frequency response.

In Section 1 the Fourier inversion integral for the steady-state double frequency response is obtained. The primitive inversion contour is deformed in the k plane to a contour having the general character of the path of steepest descents for the ion integral in the linear problem. Section 2 describes the transformation of the variable of integration from k to $\zeta = 2\omega_o/ka_i$ and the separation of the integral into ion-like and electron-like integrals. Sections 3, 4, and 5 describe the treatment of important numerical considerations in the evaluation of the ion-like integrals, of the electron-like integrals, and of the residue contributions, respectively. The calculations and numerical results are described in Section 6.

4.1 Formulation of Fourier Integrals and Deformation of Contour

The contribution of species α to the Laplace-Fourier transform of the partial excitation under consideration is denoted by $S_\alpha^{(+)}(k,v,\omega) = T[e^{-2i\omega_o t}\, S_\alpha^{(+)}(s,v)]$. From Equations (3.3.1) and (3.3.14), we have

$$S_\alpha^{(+)}(k,v,\omega) = f^2 \frac{i}{(\omega - 2\omega_0)} \frac{q_\alpha}{m_\alpha} AB \int_{-\infty}^{\infty} dx\, e^{-ikx + 2ik_1|x|}$$

$$\times \left\{ \theta(x) \frac{\partial}{\partial v} \left[\frac{\partial f_{o\alpha}/\partial v}{(v - \zeta_1 a_i)_+} \right] - \theta(-x) \frac{\partial}{\partial v} \left[\frac{\partial f_{o\alpha}/\partial v}{(v + \zeta_1 a_i)_-} \right] \right\} . \tag{4.1.1}$$

Evaluating the integrals, this is

$$S_\alpha^{(+)}(k,v,\omega) = f^4 \frac{i}{(\omega - 2\omega_0)} \frac{q_\alpha}{m_\alpha} \left(\frac{AB}{i} \right)$$

$$\times \left\{ \frac{\partial}{\partial v} \left[\frac{\partial f_{o\alpha}/\partial v}{(v - \zeta_1 a_i)_+} \right] \frac{1}{(k - 2k_1)} + \frac{\partial}{\partial v} \left[\frac{\partial f_{o\alpha}/\partial v}{(v + \zeta_1 a_i)_-} \right] \frac{1}{(k + 2k_1)} \right\} . \tag{4.1.2}$$

The Fourier transforms are valid throughout the k plane, except at $k = 2k_1$. Introducing this expression into Equation (3.2.4), and analytically continuing the functions defined by velocity integrals, there results, in an obvious notation,

$$\frac{\delta \phi^{(+)}(k,\omega)}{f^6} = \frac{i}{(\omega - 2\omega_0)} \frac{AB}{k^3 \left[f^2 K(k,\omega) \right]}$$

$$\times \sum_\alpha U_\alpha \left\{ \frac{V_{\pm +}\left(\frac{\omega_0}{k a_\alpha}, \mu_\alpha^{1/2} \zeta_1 \right)}{(k - 2k_1)} + \frac{V_{\pm -}\left(\frac{\omega_0}{k a_\alpha}, -\mu_\alpha^{1/2} \zeta_1 \right)}{(k + 2k_1)} \right\} . \tag{4.1.3}$$

Accordingly, the Fourier inversion integral from which the double frequency response may be obtained according to Equation (3.3.17) is

$$\frac{S^{(+)}(x)}{f^6} = AB\sum_\alpha U_\alpha\left\{\int_{-\infty}^{0}\frac{dk}{2\pi}\frac{e^{ikx}V_{-+}\left(\frac{2\omega_0}{ka_\alpha},\mu_\alpha^{1/2}\zeta_1\right)}{k^3[f^2K_-(k,2\omega_0)](k-2k_1)}\right.$$

$$+\int_{0}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}V_{++}\left(\frac{2\omega_0}{ka_\alpha},\mu_\alpha^{1/2}\zeta_1\right)}{k^3[f^2K_+(k,2\omega_0)](k-2k_1)}$$

$$+\int_{-\infty}^{0}\frac{dk}{2\pi}\frac{e^{ikx}V_{--}\left(\frac{2\omega_0}{ka_\alpha},-\mu_\alpha^{1/2}\zeta_1\right)}{k^3[f^2K_-(k,2\omega_0)](k+2k_1)}$$

$$\left.+\int_{0}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}V_{+-}\left(\frac{2\omega_0}{ka_\alpha},-\mu_\alpha^{1/2}\zeta_1\right)}{k^3[f^2K_+(k,2\omega_0)](k+2k_1)}\right\}. \qquad (4.1.4)$$

The evaluation of these integrals by methods similar to those used in the linearized problem is now performed. They will be expressed as the sum of a residue contribution, of ion-like integrals, and of electron-like integrals. By an ion-like (electron-like) integral is meant an integral whose integrand is a product of a factor $e^{2iz/\zeta-\zeta^2}$ ($e^{2iz/\zeta-\mu\zeta^2}$) and a function not exhibiting exponential behavior.

We consider positive values of x. In each of the four integrals (for given α) of Equation (4) the same function $Z'_\pm\left(\frac{2\omega_o}{ka_\alpha}\right)$ appears both in V and in K. Therefore, the relation of the primitive Fourier inversion contour to branch cuts is the same as in the linearized problem. See Figure 3. Because $V_{-+}\left(\frac{2\omega_o}{ka_\alpha}, \mu_\alpha{}^{1/2}\zeta_1\right)$ has a pole at $k = 2k_1$, we proceed in the k plane directly to the deformed contour shown in Figure 9. For $z \gtrsim 1.9$, this contour has the general character of the path of steepest descents for an ion-like integral.

We consider the integrals whose primitive contour of integration is the negative real axis. The behavior of $V_{-\pm}\left(\frac{2\omega_o}{ka_\alpha}, \pm\mu_\alpha^{1/2}\zeta_1\right)$ as $|k| \to 0$, $0 \leq \arg\{k\} \leq \pi$, must be determined. In this region, the asymptotic behavior

$$Z_-'(\zeta) \sim \zeta^{-2} \qquad Z_-''(\zeta) \sim -2\zeta^{-3} \tag{4.1.5}$$

gives

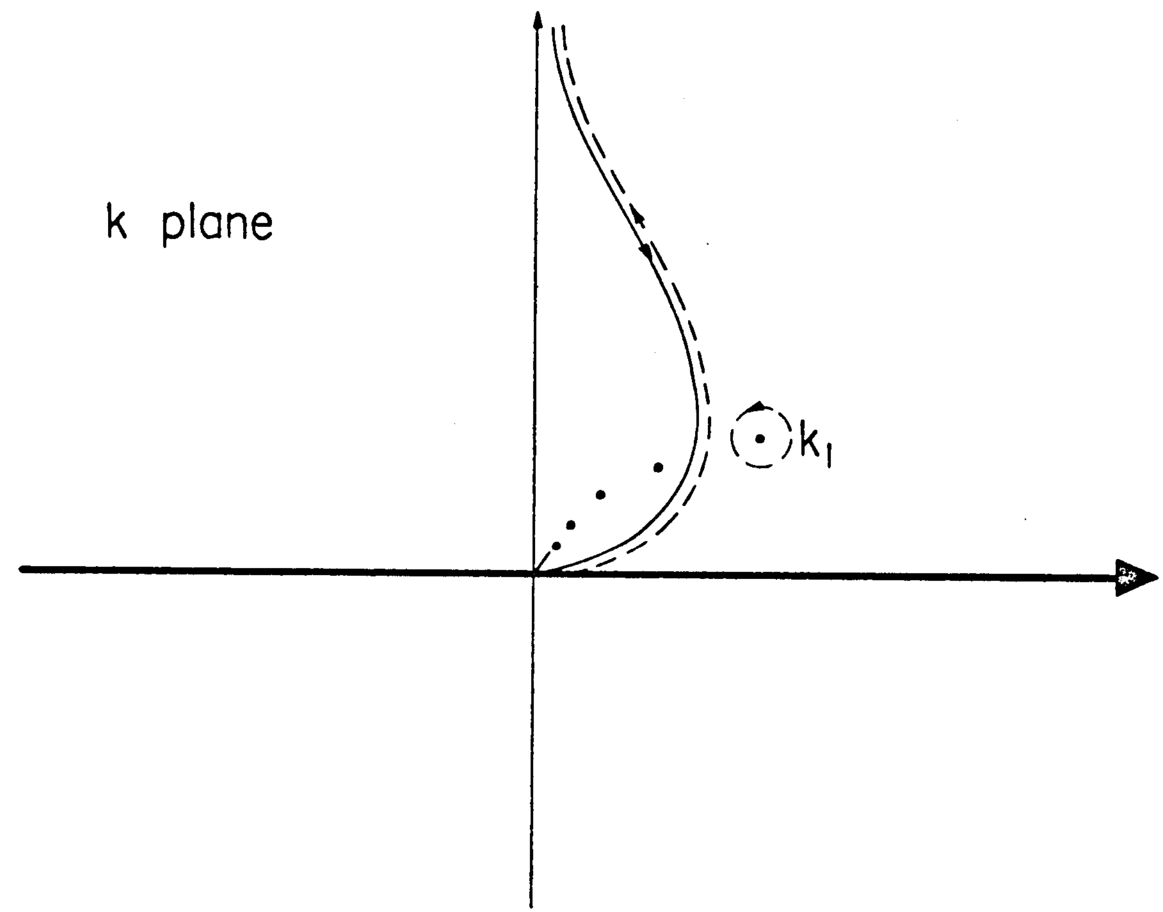
$$V_{-\pm}\left(\zeta_\alpha, \pm\mu_\alpha^{1/2}\zeta_\alpha\right) \sim \frac{Z_\pm'\left(\pm\mu_\alpha^{1/2}\zeta_1\right)}{\zeta^2} \propto k^2 . \tag{4.1.6}$$

The behavior of $V_{+\pm}(\zeta_\alpha, \pm\mu_\alpha^{1/2}\zeta_\alpha)$ as $k \to 0$ on the positive axis is the same. Hence, there is a simple pole of the integrals at $k = 0$, as in the linearized case. It has no effect on the calculation after the folding of the contour in the k plane.

As $|k| \to \infty$, $V_{\pm(\;)}(\zeta,\zeta') \to$ constant. Hence, the deformation of contour described can be carried out. Denoting the deformed contour by C, we have

$$\frac{S^{(4)}(x)}{f^6} = AB\sum_\alpha U_\alpha\left\{\int_C \frac{dk}{2\pi}\frac{e^{ikx}}{k^3(k-2k_1)}\left[\frac{V_{++}\left(\frac{2\omega_0}{ka_\alpha}, \mu_\alpha^{1/2}\zeta_1\right)}{f^2K_+(k,2\omega_0)} - \frac{V_{-+}\left(\frac{2\omega_0}{ka_\alpha}, \mu_\alpha^{1/2}\zeta_1\right)}{f^2K_-(k,2\omega_0)}\right]\right.$$

$$\left.+\int_C\frac{dk}{2\pi}\frac{e^{ikx}}{k^3(k+2k_1)}\left[\frac{V_{+-}\left(\frac{2\omega_0}{ka_\alpha}, -\mu_\alpha^{1/2}\zeta_1\right)}{f^2K_+(k,2\omega_0)} - \frac{V_{--}\left(\frac{2\omega_0}{ka_\alpha}, -\mu_\alpha^{1/2}\zeta_1\right)}{f^2K_-(k,2\omega_0)}\right]\right\} + P . \tag{4.1.7}$$



FOLDED CONTOUR OF INTEGRATION IN k PLANE FOR INTEGRALS IN NONLINEAR THEORY.

FIGURE 9.

The character of the residue contribution, P , will be considered below.

4.2 Transformation of Variable of Integration and Separation of Integrals

We now make a transformation of the variable of integration to $\zeta \equiv 2\omega_o/ka_i$. In terms of this variable,*

$$\frac{1}{k \mp 2k_1} = \mp \frac{a_i}{2\omega_o} \zeta_1 \frac{\zeta}{\zeta \mp \zeta_1} . \tag{4.2.1}$$

For the case $f^2 << 1$ ($f = \omega_o/\omega_{pi}$, as before) to which we now limit consideration,

$$f^2 K_{\pm}(k, 2\omega_o) \longrightarrow \frac{1}{4} f^2 K_{\pm}(\zeta) . \tag{4.2.2}$$

As noted in treating the zero frequency component, for a plasma composed of electrons and singly charged ions at the same temperature, $U_i = U = - U_e$, where U is given by Equation (3.5.16). The definition of the dimensionless distance, z , remains unchanged. Making use of these relations, there results

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* The relation between ζ_1 and k_1 is unchanged: $\zeta_1 = \omega_o/k_1 a_i$. Also, as before, $\zeta_1 \simeq 1.45 - 0.60i$.

$$\frac{S^{(+)}(x)}{f^6} = C\int_{C'} d\zeta\, \zeta^2 e^{\frac{2iz}{\zeta}} \left\{ \frac{1}{(\zeta-\zeta_1)} \left[\frac{V_{++}(\zeta,\zeta_1)}{f^2K_+(\zeta)} - \frac{V_{-+}(\zeta,\zeta_1)}{f^2K_-(\zeta)} \right.\right.$$

$$\left. - \frac{V_{++}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_1)}{f^2K_+(\zeta)} + \frac{V_{-+}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_1)}{f^2K_-(\zeta)} \right] - \frac{1}{(\zeta+\zeta_1)} \left[\frac{V_{+-}(\zeta,-\zeta_1)}{f^2K_+(\zeta)} \right.$$

$$\left.\left. - \frac{V_{--}(\zeta,-\zeta_1)}{f^2K_-(\zeta)} - \frac{V_{+-}(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_1)}{f^2K_+(\zeta)} + \frac{V_{--}(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_1)}{f^2K_-(\zeta)} \right] \right\} + P. \qquad (4.2.3)$$

in which C' denotes the contour of Figure 6, and

$$C = \frac{i}{16\pi}\left(\frac{\sigma_0 x_0}{\epsilon_0}\right)^2 \left(\frac{a_i}{\omega_0}\right)^2 \frac{U}{\zeta_1^2[f^2K_+'(\zeta_1)]^2} \quad . \qquad (4.2.4)$$

The presence of the V's prevents a direct resolution into electron-like and ion-like integrals of the sort achieved in Equation (2.2.7) through the use of Equation (2.2.6). We make use of the relation

$$V_{-\pm}(\zeta,\pm\zeta') = V_{+\pm}(\zeta,\pm\zeta') + 4\sqrt{\pi}\, i\, R(\zeta,\pm\zeta')\, e^{-\zeta^2} \qquad (4.2.5)$$

where

$$R(\zeta,\zeta') = \left[\frac{-\zeta + (1-2\zeta^2)(\zeta-\zeta')}{(\zeta-\zeta')^2} \right] \qquad (4.2.6)$$

which follows from Equations (3.3.7) and (3.4.7). With this relation and Equation (2.2.6) we obtain the desired resolution:

$$\frac{S^{(+)}(x)}{f^{6}(-2\pi i C)} = \frac{2}{\sqrt{\pi}} \int_{C'} \frac{d\zeta\, e^{\frac{2iz}{\zeta}} \left(e^{-\zeta^2} + \mu^{1/2} e^{-\mu\zeta^2}\right)}{\zeta^{-2}\left[f^2 K_+(\zeta)\right]\left[f^2 K_-(\zeta)\right]} \left\{ \zeta^3 \left[\frac{V_{++}(\zeta,\zeta_1)}{(\zeta-\zeta_1)} - \frac{V_{+-}(\zeta,-\zeta_1)}{(\zeta+\zeta_1)} \right] \right.$$

$$\left. - \zeta^3 \left[\frac{V_{++}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_1)}{(\zeta-\zeta_1)} - \frac{V_{+-}(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_1)}{(\zeta+\zeta_1)} \right] \right\}$$

$$+ \frac{2}{\sqrt{\pi}} \int_{C'} \frac{d\zeta\, e^{\frac{2iz}{\zeta} - \zeta^2}}{\zeta^{-2}\left[f^2 K_-(\zeta)\right]} \left[\frac{R(\zeta,\zeta_1)}{(\zeta-\zeta_1)} - \frac{R(\zeta,-\zeta_1)}{(\zeta+\zeta_1)} \right]$$

$$- \frac{2}{\sqrt{\pi}} \int_{C'} \frac{d\zeta\, e^{\frac{2iz}{\zeta} - \mu\zeta^2}}{\zeta^{-2}\left[f^2 K_-(\zeta)\right]} \left[\frac{R(\mu^{1/2}\zeta,\mu^{1/2}\zeta_1)}{(\zeta-\zeta_1)} - \frac{R(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_1)}{(\zeta+\zeta_1)} \right] + P' \qquad (4.2.7)$$

where $P' = P/(-2\pi i C)$.

The first term consists of the sum of an ion-like integral and an electron-like integral whose integrands differ in the same way as the integrands of the ion and electron integrals in the linear response. That is, the integrands differ only in that one contains the factor $e^{-\zeta^2}$ while the other contains the factor $\mu^{1/2} e^{-\mu\zeta^2}$. See Equation (2.2.7). If the curly brackets contained unity, the first term would be the potential in the linearized theory at the point $2z$. The second and third terms are cast in a form which resembles, as far as is feasible, the integrals in the linearized theory. A rearrangement of the integrals which is convenient from a computational standpoint is

$$
\begin{aligned}
\frac{S^{(+)}(x)}{f^{6}(-2\pi i C)} = \frac{2}{\sqrt{\pi}} \int_{C'} \frac{d\zeta\, e^{\frac{2iz}{\zeta} - \zeta^{2}}}{\zeta^{-2}[f^{2}K_{+}(\zeta)][f^{2}K_{-}(\zeta)]} \Bigg\{ & \zeta^{3}\left[\frac{V_{++}(\zeta,\zeta_{1})}{(\zeta-\zeta_{1})} - \frac{V_{+-}(\zeta,-\zeta_{1})}{(\zeta+\zeta_{1})}\right] \\
& + [f^{2}K_{+}(\zeta)]\left[\frac{R(\zeta,\zeta_{1})}{(\zeta-\zeta_{1})} - \frac{R(\zeta,-\zeta_{1})}{(\zeta+\zeta_{1})}\right] - \zeta^{3}\left[\frac{V_{++}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_{1})}{(\zeta-\zeta_{1})} - \frac{V_{+-}(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_{1})}{(\zeta+\zeta_{1})}\right]\Bigg\} \\
+ \mu^{1/2}\frac{2}{\sqrt{\pi}} \int_{C'} \frac{d\zeta\, e^{\frac{2iz}{\zeta} - \mu\zeta^{2}}}{\zeta^{-2}[f^{2}K_{+}(\zeta)][f^{2}K_{-}(\zeta)]} \Bigg\{ & \zeta^{3}\left[\frac{V_{++}(\zeta,\zeta_{1})}{(\zeta-\zeta_{1})} - \frac{V_{+-}(\zeta,-\zeta_{1})}{(\zeta+\zeta_{1})}\right] \\
& - \zeta^{3}\left[\frac{V_{++}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_{1})}{(\zeta-\zeta_{1})} - \frac{V_{+-}(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_{1})}{(\zeta+\zeta_{1})}\right]\Bigg\} \\
+ \frac{2}{\sqrt{\pi}} \int_{C'} \frac{d\zeta\, e^{\frac{2iz}{\zeta} - \mu\zeta^{2}}}{\zeta^{-2}[f^{2}K_{-}(\zeta)]} \Bigg\{ & -\left[\frac{R(\mu^{1/2}\zeta,\mu^{1/2}\zeta_{1})}{(\zeta-\zeta_{1})} - \frac{R(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_{1})}{(\zeta+\zeta_{1})}\right]\Bigg\} + P' . \qquad (4.2.8)
\end{aligned}
$$

This expression for $S^{(+)}(x)$ in the dominant pole approximation will be evaluated numerically below. The double frequency component of the lowest order nonlinear response is given by $2\mathrm{Re}\{e^{-2i\omega_0 t}\, S^{(+)}(x)\}$. See Equation (3.3.17).

4.3 Evaluation of Ion-like Integral

The path of steepest descents for the integrand $e^{2iz/\zeta-\zeta^2}$ is chosen for the evaluation of the ion-like integral. Since the dominant pole approximation is probably not valid for $z \lesssim 2.5$, as is discussed below, the steepest descent contour lies below the first pole for all values of z which are of interest. Certain important numerical considerations are now discussed.

It is convenient to treat the parts of the ion-like integral which contain $V_{+\pm}(\zeta,\pm\zeta_1)$ and $R(\zeta,\pm\zeta_1)$ as a unit. In preparation for this, we state the identity

$$Z''_{\pm}(\zeta) = \frac{2}{\zeta} + \frac{1}{\zeta}(1-2\zeta^2)Z'_{\pm}(\zeta)\ , \tag{4.3.1}$$

which may be obtained from Equation (3.3.7). Using this identity and Equation (3.3.9) we obtain the two relations

$$V_{++}(\zeta,\zeta_1) = \frac{Z'_+(\zeta_1)}{(\zeta-\zeta_1)^2} + \frac{2}{\zeta(\zeta-\zeta_1)} + \frac{1}{\zeta}R(\zeta,\zeta_1)Z'_+(\zeta) \tag{4.3.2}$$

$$V_{+-}(\zeta,-\zeta_1) = \frac{Z'_+(\zeta_1)}{(\zeta+\zeta_1)^2} + \frac{2}{\zeta(\zeta+\zeta_1)} + \frac{1}{\zeta}R(\zeta,-\zeta_1)Z'_+(\zeta). \tag{4.3.3}$$

Application of Equation (2) gives

$$\frac{V_{++}(\zeta,\zeta_1)}{(\zeta-\zeta_1)} + \frac{f^2K_+(\zeta)}{\zeta^3}\frac{R(\zeta,\zeta_1)}{(\zeta-\zeta_1)} = \frac{Z'_+(\zeta_1)}{(\zeta-\zeta_1)^3} + \frac{2}{\zeta(\zeta-\zeta_1)^2} - \frac{Z'_+(\mu^{1/2}\zeta)R(\zeta,\zeta_1)}{\zeta(\zeta-\zeta_1)}\ . \tag{4.3.4}$$

Similarly, from Equation (3), we have

$$\frac{V_{+-}(\zeta,\zeta_1)}{(\zeta+\zeta_1)} + \frac{f^2K_+(\zeta)}{\zeta^3}\frac{R(\zeta,-\zeta_1)}{(\zeta+\zeta_1)} = \frac{Z'_+(\zeta_1)}{(\zeta+\zeta_1)^3} + \frac{2}{\zeta(\zeta+\zeta_1)^2} - \frac{Z'_+(\mu^{1/2}\zeta)R(\zeta,-\zeta_1)}{\zeta(\zeta+\zeta_1)}\ . \tag{4.3.5}$$

Combining these results and using the relation $Z'_+(\zeta_1) = -Z'_+(\mu^{1/2}\zeta_1)$, which results from the fact that ζ_1 is a solution of the dispersion relation $f^2K_+(\zeta_1) = 0$ [see Equation (2.2.8)], we have

$$\left[\frac{V_{++}(\zeta,\zeta_1)}{(\zeta-\zeta_1)} - \frac{V_{+-}(\zeta,-\zeta_1)}{(\zeta+\zeta_1)}\right] + \frac{[F^2K_+(\zeta)]}{\zeta^3}\left[\frac{R(\zeta,\zeta_1)}{(\zeta-\zeta_1)} - \frac{R(\zeta,-\zeta_1)}{(\zeta+\zeta_1)}\right]$$

$$= \frac{1}{\zeta}\left[2+(2\zeta^2-1)Z'_+(\mu^{1/2}\zeta)\right]\left[\frac{1}{(\zeta-\zeta_1)^2} - \frac{1}{(\zeta+\zeta_1)^2}\right] + \left[Z'_+(\mu^{1/2}\zeta) - Z'_+(\mu^{1/2}\zeta_1)\right]\left[\frac{1}{(\zeta-\zeta_1)^3} - \frac{1}{(\zeta+\zeta_1)^3}\right] . \quad (4.3.6)$$

Since $|\zeta| \lesssim O(1)$ for the ion-like integral, it is necessary to use an approximation based upon the power series expansion

$$\left[Z'_+(\mu^{1/2}\zeta) - Z'_+(\mu^{1/2}\zeta_1)\right] = \mu^{1/2} Z''_+(\mu^{1/2}\zeta_1)(\zeta-\zeta_1) + \frac{\mu}{2!} Z'''_+(\mu^{1/2}\zeta_1)(\zeta-\zeta_1)^2 + \cdots . \quad (4.3.7)$$

We now consider the part of the ion-like integral which contains $V_{+\pm}(\mu^{1/2}\zeta, \pm\mu^{1/2}\zeta_1)$. The function $V_{++}(\mu^{1/2}\zeta, \mu^{1/2}\zeta_1)$ is regular at $\zeta = \zeta_1$, as may be verified by expanding $Z'_+(\mu^{1/2}\zeta)$ and $Z''_+(\mu^{1/2}\zeta)$ in power series around $\mu^{1/2}\zeta_1$ and substituting into the appropriate form of Equation (3.4.7). The result is

$$V_{++}(\mu^{1/2}\zeta, \mu^{1/2}\zeta_1) = \frac{1}{2} Z'''_+(\mu^{1/2}\zeta_1) + \frac{\mu^{1/2}}{3} Z^{(iv)}_+(\mu^{1/2}\zeta_1)(\zeta-\zeta_1) + \cdots . \quad (4.3.8)$$

It is used to approximate $V_{++}(\mu^{1/2}\zeta, \mu^{1/2}\zeta_1)$ in the ion-like integral. The function $V_{+-}(\mu^{1/2}\zeta, -\mu^{1/2}\zeta_1)$ is conveniently expressed for computational purposes in the ion-like integral as

$$V_{+-}(\mu^{1/2}\zeta, -\mu^{1/2}\zeta_1) = \frac{Z''_+(\mu^{1/2}\zeta_1)}{\mu^{1/2}(\zeta+\zeta_1)} + \frac{\left[Z''_+(\mu^{1/2}\zeta) - Z''_+(\mu^{1/2}\zeta_1)\right]}{\mu^{1/2}(\zeta+\zeta_1)} + \frac{\left[Z'_+(\mu^{1/2}\zeta_1) - Z'_+(\mu^{1/2}\zeta)\right]}{\mu(\zeta+\zeta_1)^2} . \quad (4.3.9)$$

The second and third terms consist of the product of a large factor ($\mu^{-1/2}$ and μ^{-1}, respectively) and a small factor. The computational difficulty is removed by expanding $Z''_+(\mu^{1/2}\zeta)$ and $Z'_+(\mu^{1/2}\zeta)$ in power series around $\mu^{1/2}\zeta_1$. The result is

$$V_{+-}(\mu^{1/2}\zeta, -\mu^{1/2}\zeta_1) = \frac{Z''_+(\mu^{1/2}\zeta_1)}{\mu^{1/2}(\zeta+\zeta_1)}\left[1 - \left(\frac{\zeta-\zeta_1}{\zeta+\zeta_1}\right)\right] + \frac{Z'''_+(\mu^{1/2}\zeta_1)}{(\zeta+\zeta_1)}(\zeta-\zeta_1)\left[1 - \frac{1}{2}\left(\frac{\zeta-\zeta_1}{\zeta+\zeta_1}\right)\right]$$

$$+ \frac{\mu^{1/2} Z^{(iv)}_+(\mu^{1/2}\zeta_1)}{2!(\zeta+\zeta_1)}(\zeta-\zeta_1)^2\left[1 - \frac{1}{3}\left(\frac{\zeta-\zeta_1}{\zeta+\zeta_1}\right)\right] + \frac{\mu Z^{(v)}_+(\mu^{1/2}\zeta_1)}{3!(\zeta+\zeta_1)}(\zeta-\zeta_1)^3\left[1 - \frac{1}{4}\left(\frac{\zeta-\zeta_1}{\zeta+\zeta_1}\right)\right] + \cdots . \quad (4.3.10)$$

The relations derived in this section make it possible for us to evaluate the ion-like integral of Equation (4.2.8) along the path of steepest descents through the appropriate saddle point of the exponent in the function $e^{2iz/\zeta-\zeta^2}$.

4.4 Evaluation of Electron-like Integral

The part of the electron-like integral in Equation (4.2.8) containing a factor $\mu^{1/2}$ bears the same relation to parts of the ion-like integral that the electron integral bears to the ion integral in the linearized problem. Therefore, numerical integration of this integral may be started at a value of ζ for which $|\zeta|$ is sufficiently large that asymptotic expansions may be used to approximate derivatives of $Z_+(\zeta)$ in the integrand. The deformed contour of integration used is that chosen for the electron integral in the linearized problem with $z \to 2z$. The quantity in square brackets containing $V_{+\pm}(\zeta,\pm\zeta_1)$ is expressed as

$$\left[\frac{V_{++}(\zeta,\zeta_1)}{(\zeta-\zeta_1)} - \frac{V_{+-}(\zeta,-\zeta_1)}{(\zeta+\zeta_1)}\right]$$

$$= \left[Z_+'(\zeta_1) - Z_+'(\zeta)\right]\left[\frac{2\zeta_1(3\zeta^2+\zeta_1^2)}{(\zeta^2-\zeta_1^2)^3}\right] + Z_+''(\zeta)\left[\frac{4\zeta_1\zeta}{(\zeta^2-\zeta_1^2)^2}\right] . \tag{4.4.1}$$

In the asymptotic region, this is approximately equal to $(12\zeta_1/\zeta^4)$. Evaluation of the integral containing the factor of Equation (1), based on this approximation, shows that it makes a negligible contribution to a three-place calculation of $[S^{(+)}(x)/f^6(-2\pi iC)]$. The quantity in square brackets containing $V_{+\pm}(\mu^{1/2}\zeta,\pm\mu^{1/2}\zeta_1)$ is expressed in a form convenient both for computation and for inspection of its behavior as $|\zeta|$ becomes large, as

$$\left[\frac{V_{++}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_1)}{(\zeta-\zeta_1)} - \frac{V_{+-}(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_1)}{(\zeta+\zeta_1)}\right] = \frac{4\zeta_1}{\mu^{1/2}\zeta^3}\Bigg\{\left[Z_+(\mu^{1/2}\zeta) - \left(\frac{\zeta_1}{\zeta}\right)Z_+(\mu^{1/2}\zeta_1)\right]$$

$$\times\frac{[3+(\zeta_1/\zeta)^2]}{[1-(\zeta_1/\zeta)^2]^3} - 2\left[Z_+(\mu^{1/2}\zeta) + (\mu^{1/2}\zeta)Z_+'(\mu^{1/2}\zeta)\right]\frac{1}{[1-(\zeta_1/\zeta)^2]^2}\Bigg\} . \tag{4.4.2}$$

The evaluation of the remaining electron-like integral of Equation (4.2.8) is now considered. This integral differs from the other electron-like integral in not having a counterpart in the ion-like integral. Hence, it is not possible here to start the numerical integration in the asymptotic region for plasma dispersion functions of argument ζ . For evaluation, the quantity in curly brackets, which we denote by N , must be expressed in two forms for $|\zeta|$ less than and greater than $O(1)$. These forms are

$$N_{<}=-\frac{2\zeta}{\mu^{1/2}\zeta_1^{3}\left[1-(\zeta/\zeta_1)^2\right]^2}\left\{\frac{\left[1+3(\zeta/\zeta_1)^2\right]}{\left[1-(\zeta/\zeta_1)^2\right]}+2(1-2\mu\zeta^2)\right\} \quad (4.4.3)$$

$$N_{>}=-\frac{2\zeta_1}{\mu^{1/2}\zeta^{3}\left[1-(\zeta_1/\zeta)^2\right]^2}\left\{-\frac{\left[3+(\zeta_1/\zeta)^2\right]}{\left[1-(\zeta_1/\zeta)^2\right]}+2(1-2\mu\zeta^2)\right\}. \quad (4.4.4)$$

The absence of a factor $[f^2K_+(\zeta)]$ in the integral being considered permits us to use as a contour of integration a straight line from the origin with an angle to the positive real axis of $-\pi/6$.

4.5 Residue Contributions

The residue contributions to $[S^{(+)}(x)/f^6(-2\pi iC)]$ may be determined from Equation (4.2.3). There is a residue contribution from each term which contains $[f^2K_+(\zeta)]$. The residue contributions in Equation (4.2.7) are given by

$$P'=\mathrm{Res}\left[\frac{\zeta^2 e^{\frac{2iz}{\zeta}}}{f^2K_+(\zeta)}\left\{\frac{1}{(\zeta-\zeta_1)}\left[V_{++}(\zeta,\zeta_1)-V_{++}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_1)\right]-\frac{1}{(\zeta+\zeta_1)}\left[V_{+-}(\zeta,-\zeta_1)-V_{+-}(\mu^{1/2}\zeta,-\mu^{1/2}\zeta_1)\right]\right\}\right], \quad (4.5.1)$$

The first group of terms has a double pole at $\zeta = \zeta_1$, the second a simple pole.

To obtain the residue of the double pole, each factor in the first group of terms is expanded in a Laurent series around $\zeta = \zeta_1$. The Laurent series for $[f^2 K_+(\zeta)]^{-1}$ is represented by

$$\frac{1}{f^2 K_+(\zeta)} = \frac{a_{-1}}{(\zeta - \zeta_1)} + a_0 + \cdots \tag{4.5.2}$$

where $a_{-1} = f^2 K'_+(\zeta_1)$ and

$$a_0 = \lim_{\zeta \to \zeta_1} \frac{d}{d\zeta} \left\{ \left[\frac{f^2 K_+(\zeta)}{(\zeta - \zeta_1)} \right]^{-1} \right\} = -\frac{1}{2} \frac{[f^2 K''_+(\zeta_1)]}{[f^2 K'_+(\zeta_1)]^2} . \tag{4.5.3}$$

The expansions for ζ^2, $e^{2iz/\zeta}$, and $[V_{++}(\zeta,\zeta_1) - V_{++}(\mu^{1/2}\zeta,\mu^{1/2}\zeta_1)]$ [see Equation (4.3.8)], are

$$\zeta^2 = \zeta_1^2 + 2\zeta_1(\zeta - \zeta_1) + (\zeta - \zeta_1)^2 \tag{4.5.4}$$

$$e^{\frac{2iz}{\zeta}} = e^{\frac{2iz}{\zeta_1}} - \frac{2iz}{\zeta_1^2} e^{\frac{2iz}{\zeta_1}} (\zeta - \zeta_1) + \cdots \tag{4.5.5}$$

$$[V_{++}(\zeta,\zeta_1) - V_{++}(\mu^{1/2}\zeta, \mu^{1/2}\zeta_1)]$$

$$= \frac{1}{2}\left[Z'''_+(\zeta_1) - Z'''_+(\mu^{1/2}\zeta_1)\right] + \frac{1}{3}\left[Z^{(iv)}_+(\zeta_1) - \mu^{1/2} Z^{(iv)}_+(\mu^{1/2}\zeta_1)\right](\zeta - \zeta_1) + \cdots . \tag{4.5.6}$$

The residue of the double pole is the coefficient of $(\zeta - \zeta_1)^{-1}$ in the product of the Laurent expansions. The total residue is

$$
\begin{aligned}
P' = \frac{\zeta_1^2 e^{\frac{2iz}{\zeta_1}}}{f^2 K_+'(\zeta_1)} \Bigg[& \frac{1}{2}\left[Z_+'''(\zeta_1) - Z_+'''(\mu^{1/2}\zeta_1)\right] \\
& \times \left\{ \left[\frac{2}{\zeta_1} - \frac{1}{2}\frac{f^2 K_+''(\zeta_1)}{f^2 K_+'(\zeta_1)}\right] + \frac{2}{3}\left[\frac{Z_+^{(iv)}(\zeta_1) - \mu^{1/2} Z_+^{(iv)}(\mu^{1/2}\zeta_1)}{Z_+'''(\zeta_1) - Z_+'''(\mu^{1/2}\zeta_1)}\right] - \frac{2iz}{\zeta_1^2} \right\} \\
& - \frac{1}{2\zeta_1}\left[V_{+-}(\zeta_1,-\zeta_1) - V_{+-}(\mu^{1/2}\zeta_1, -\mu^{1/2}\zeta_1)\right] \Bigg] .
\end{aligned}
\tag{4.5.7}
$$

The V's are determined by

$$
V_{+-}(\zeta,-\zeta) = \frac{1}{2\zeta^2}\left[2 + (1 - 2\zeta^2) Z_+'(\zeta)\right] . \tag{4.5.8}
$$

4.6 Numerical Results

The range of z for which the dominant pole approximation yields a valid prediction of the lowest order nonlinear response is now estimated. The substantial deviations of the potential in the linear theory from the dominant pole approximation are two: the large amplitude, heavily damped, contribution near the grid, and the electron wave at large distance, which is weakly damped.

Let each of these two contributions be represented approximately by an exponential term with a complex wave number. Each produces a contribution to $S_\alpha^{(+)}(k,v,\omega)$ of the same form as Equation (4.1.2) with an appropriate amplitude and with a pair of simple poles at $\pm 2k_g$ or $\pm 2k_e$. The subscripts g and e denote the heavily damped response near the grid and the electron wave, respectively. The quadratic combination of the linear response which drives $S^{(+)}(x)$ has a wave number spectrum characterized by poles at two times those in the linear response. Accordingly, since the heavily damped response near the grid gives a negligible contribution to the linear response for $z \gtrsim 5$, we expect that neglect of it in calculation of $S^{(+)}(x)$ leads to negligible error for $z \gtrsim 2.5$. Furthermore, since the electron wave produces deviations from the dominant pole response in the linear theory for $z \gtrsim 20$, we expect that calculation of $S^{(+)}(x)$ on the basis of the dominant pole approximation is not valid for $z \gtrsim 10$.

The numerical evaluation of the nonlinear response was performed on the TRW On-line Computer.[8] The results are displayed in Figure 10. No simple interpretation in terms of a dominant pole is possible. The dominant integral for all values of z is the electron-like integral of Equation (4.2.8) which does not contain the factor $\mu^{1/2}$. The integrand decreases more rapidly with increasing $|\zeta|$ than the electron integral in the linear theory and becomes very small long before the absolute value of the factor $e^{-\mu\zeta^2}$ starts to decrease substantially below unity. Thus, the response is damped much more strongly than the electron wave in the linear theory. For small values of z the residue contributions are comparable with the integral just discussed; by $z \approx 7$, they become negligible. An interesting characteristic of the response is that, although the damping is sufficiently strong to be characteristic of an ion wave, the change in phase is so small as to suggest an electron wave.

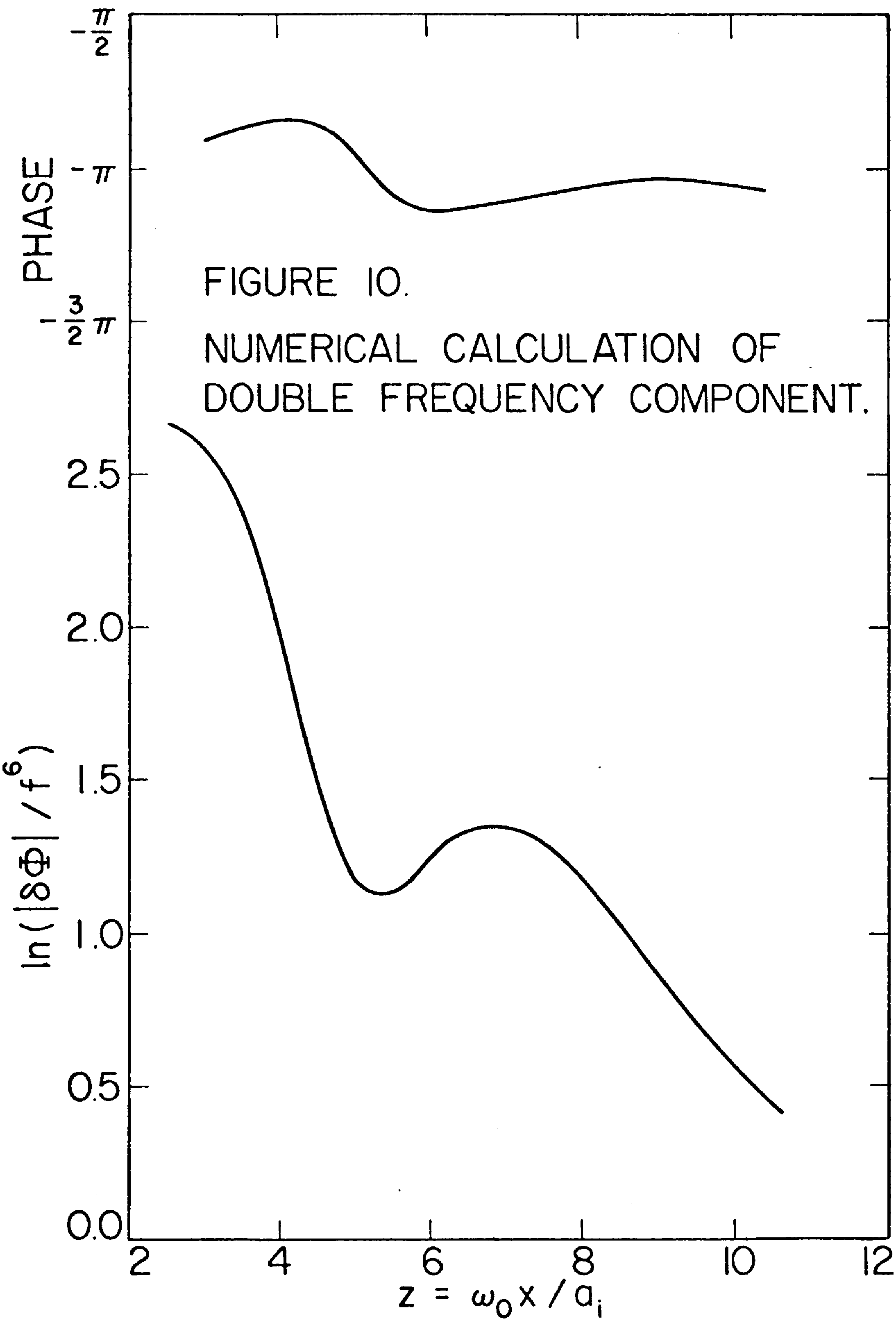

FIGURE 10.

NUMERICAL CALCULATION OF DOUBLE FREQUENCY COMPONENT.

V. CONCLUSION

We have introduced a new method for the numerical evaluation of the Fourier inversion integral which occurs in the linearized theory of grid excitation of low frequency longitudinal waves in a correlationless plasma. The advantages of the method, discussed above, lead us to view the variable $\zeta = \omega_o/ka_i$ as more appropriate than the wave number, k , for treating this problem. The numerical results obtained for the case of finite separation between the grids demonstrate that the nature of the response is determined primarily by the characteristics of the plasma, as embodied in the dielectric function, rather than by the spatial characteristics of the excitation. In the dipole limit, we have verified the results obtained by Gould.[4]

We have developed a perturbation treatment of the Vlasov equation which is appropriate for the determination of the steady-state reponse of a plasma to grid excitation of amplitude somewhat greater than that for which the linear theory is appropriate. In the lowest order of the theory there are components at zero frequency and at twice the applied frequency. The zero frequency component is a static polarization of the plasma. There are no zero frequency species particle current densities. This result is independent of the perturbation expansion.

The double frequency component involves a double integral with respect to Fourier transform variables which is further complicated by the presence of branch-points in the plane of one Fourier transform variable whose position depends on the value of the other Fourier transform variable. The evaluation of the double frequency response is rendered feasible by approximating the linear response by its dominant pole contribution. A single Fourier inversion integral is obtained. It is evaluated by methods similar to those used in the linear problem. The double frequency response does not have a simple interpretation. It is strongly damped, like an "ion wave", but has a slow phase variation with distance, like an "electron wave".

Possible extensions of the present work which might shed light on the nature of the nonlinear wave include consideration of different values of the mass ratio and of the ratio of electron to ion temperature.

APPENDIX A

INVERSION OF CONFORMAL TRANSFORMATION FOR PATH OF STEEPEST DESCENTS

The inversion of the conformal transformation, Equation (2.3.4), that is, the determinations of $w(t)$, is performed by iteration. Two iterative procedures* are used; each is convergent over only part of the range $-3.1 \leq t \leq 3.1$. One procedure is based upon the variable $u = w - w_o$ and the recasting of Equation (2.3.4) as

$$u = \frac{t}{z^{1/3}} \left(\frac{u + w_o}{u + 3w_o} \right)^{1/2} . \tag{A.1}$$

The other procedure is based upon the variable $\tau = w/w_o$ and the recasting of Equation (2.3.4) as

$$\tau = \frac{2 + \tau^3}{3 + (t^2 / z^{2/3} w_o^2)} . \tag{A.2}$$

Equations (A.1) and (A.2) are of the form $v = G(v)$. They are employed to obtain successive approximations to v by the relation $v_{n+1} = G(v_n)$. The procedure of Equation (A.1) is convergent for positive values of t and for small negative values of t. The procedure of Equation (A.2) is convergent for values of t less than a small negative value. The two regions of convergence overlap for all values of z in the range of interest. In both cases, zero is a

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* These procedures were developed by Professor B. D. Fried.

satisfactory value for v_1. For iterative procedures of the type used here, little advantage in reduction of computation time is gained by a "favorable" choice of v_1, such as an approximation given by a power series in t.

The convergence characteristics of iterative procedures based on a functional relation of the fixed point[10] form $v = G(v)$ can be understood to some extent by consideration of the relation

$$v_{n+2} - v_{n+1} = G(v_{n+1}) - G(v_n). \tag{A.3}$$

Expanding $G(v_{n+1})$ in a series in powers of $(v_{n+1} - v_n)$, there results

$$\frac{v_{n+2} - v_{n+1}}{v_{n+1} - v_n} = G'(v_n) + O(v_{n+1} - v_n). \tag{A.4}$$

This suggests that an iterative procedure converges well when $|G'(v_n)|$ is small compared with unity. In the procedures used here, this rule is obeyed. Failure of convergence occurs when $|G'| = O(1)$.

The procedure of Equation (A.1) is used, with the modification of variables specified in Section 2.5, to determine the part of the path of steepest descents for $e^{iz/\zeta - \mu\zeta^2}$ which is used in evaluating the electron integral, viz., that part for which $0 \leq t \leq 3.1$.

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